Modeling the volatility of stock returns in periods of financial market stress

Christopher F Baum[A,B,1], Paola Zerilli[C]

[a]*Department of Economics, Boston College, Chestnut Hill, MA 02467 USA*
[b]*DIW Berlin, Mohrenstraße 58, 10117 Berlin*
[c]*Department of Economics and Related Studies, University of York, York YO10 5DD, UK*

Abstract

In this paper we evaluate the statistical properties of daily US stock index returns over more than three decades. The relationship between volatility of the stock return process and its fourth moments is shown to be time-varying over that horizon, which we model by identifying three subperiods with more stable properties. We find that subperiod-specific models of stochastic volatility offer superior performance to a single model fit to the entire period. During periods of high kurtosis, we propose a new model for stock returns pricing that can hold even in the event of severe financial market stress. For instance, implied volatility of the S&P500 index during 1987 was characterized by very sharp movements in both directions. Unlike existing models which only allow for dramatic increases in volatility, the proposed model allows the stock return volatility to wander about its path without any restrictions. This allows the volatility to either rise or fall violently while the entire process remains positive. The Efficient Method of Moments is used to estimate the parameters of the stock index returns process and its corresponding volatility. Improved modeling of volatility has important implications for option pricing.

Preliminary and incomplete – please do not quote

Email addresses: baum@bc.edu (Christopher F Baum), paola.zerilli@york.ac.uk (Paola Zerilli)

[1]Corresponding author.

[2]The White Rose Grid (http://academic.research.microsoft.com/Paper/3981359.aspx) was used to support this work. We would like to acknowledge the support of the White Rose Grid and White Rose Grid staff. We also thank the Department of Economics and Related Studies, University of York, for their financial support. We appreciate the comments of participants at the 18th Forecasting Financial Markets Conference, Marseille, May 2011, the 28th GdRE Symposium on Money, Banking, and Financial Markets, Reading, June 2011, the 46th Annual Conference of the Money, Macro and Finance Research Group, Birmingham, September 2011 and the 4th CSDA International Conference on Computational and Financial Econometrics, University of London, December 2010.

May 21, 2012

1. Introduction

The main objective of this paper is to construct a model for stock index returns that can hold even in the event of severe financial market stress. While stock return volatility is unobservable, a good indicator for its behavior can be found in the volatility implied in the corresponding index option market. As is evident from Figure 1, S&P500 implied volatility during the year 1987 was characterized by very sharp movements in both directions. Using econometric analysis, we split a long time period of November 1973 through December 2010 into three subperiods in which the relationship between volatility and the first, third and fourth moments of stock index returns is more stable than over the entire period. This analysis of the series' statistical properties allows us to select the model that best fits each one of these subperiods.

For the subperiod July 1986–July 1997, where stock index returns exhibit very high kurtosis, we propose a new model and show that this new model outperforms any existing structure. A high degree of kurtosis represents a much greater frequency of sizable movements, positive and negative, in the underlying series. In our data, the high kurtosis in this subperiod is accompanied by sizable differences in the mean and skewness of the series. This implies that the relationship between volatility and the first, third and fourth moments of the series does not exhibit temporal stability. Unlike existing models which only allow for dramatic increases in volatility, the proposed model allows the stock return volatility to wander about its path without any restrictions. This allows the volatility to either rise or fall violently while the entire process remains positive.

We propose a model for asset pricing that allows for spikes in volatility by constructing a new, more general log-variance model. The key aspects of the model proposed here are the following:

i) stock prices follow a mixture of Brownian motion and a multivariate compensated Poisson process;

ii) the logarithm of the variance follows an Ornstein–Uhlenbeck process with jumps whose size is random and whose sign is unrestricted;

iii) the stock price can jump both alone and together with volatility.

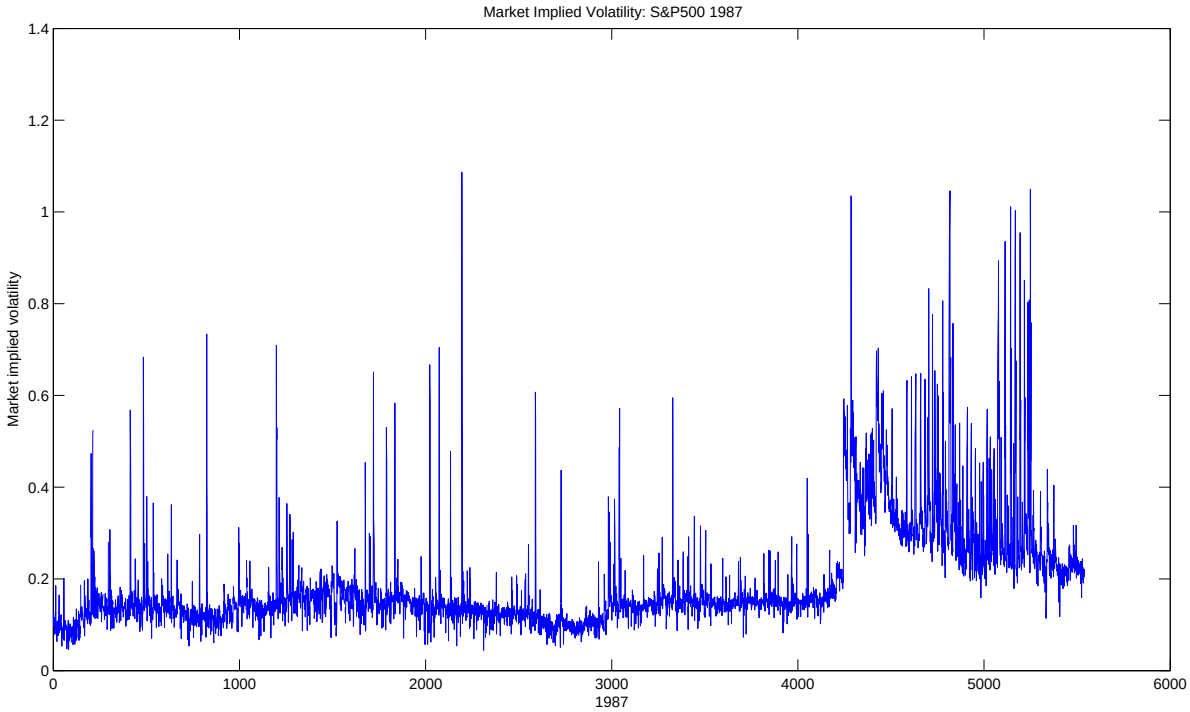

Figure 1: Spikes in the historical market implied volatility series for 1987

This last feature is based upon the empirical observation of the S&P500 option prices' at the money (ATM) one-month implied volatility (IV) and the corresponding underlying stock returns (see Figure 2). In 1987, it is very easy to notice that in some periods, stock returns and the corresponding implied volatility are moving simultaneously, while at other times there are uncorrelated movements in the two series.

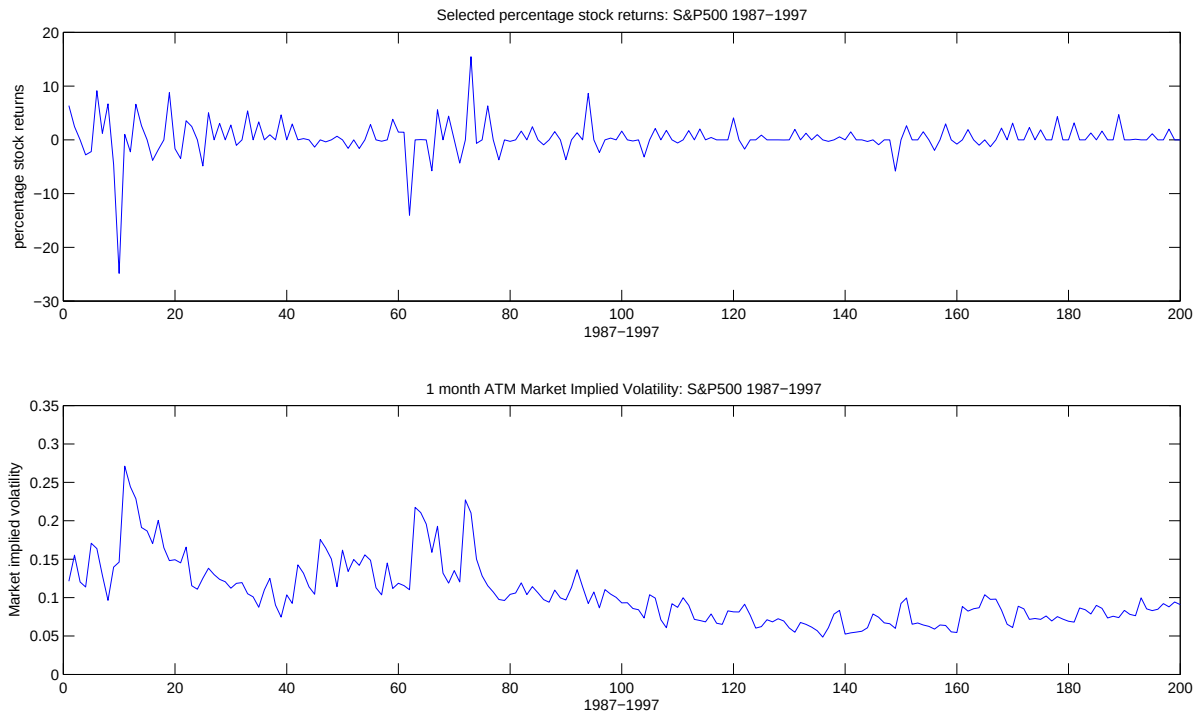

Figure 2: S&P500 at-the-money one-month implied volatility and corresponding stock returns (1987-1997).

We estimate the parameters of the structural model for the stock price dynamics using the Efficient Method of Moments (EMM) employing the time series of daily S&P500 stock returns. This choice reflects the fact that one can observe a part of the state vector (in this case, the stock return series), but not its corresponding volatility. This necessarily rules out estimation approaches such as maximum likelihood (MLE) or the generalized method of moments (GMM).

The EMM approach is based on *indirect inference*. The main idea is to replace the initial model with a more tractable approximation. The latter is denoted the *auxiliary model* and

is a descriptive model with a large number of parameters. Following Gallant & Tauchen (2002), we evaluate the scores of the auxiliary model using the simulated series of data derived from the structural model. In this way, we determine the moment conditions for this problem. The proposed log-variance model is capable of accommodating both the linear aspect and the tail behavior of the data. When estimated from the full sample, the estimate of the mean of the jumps that affect volatility is negative and significantly different from zero. This result shows that the assumption of positive jumps in volatility made by Duffie et al. (2000) is overly restrictive.

The paper is organized as follows. In Section 2, we clarify the core issues in the literature. Section 3 sets up the model and addresses the relevance of the risk premia and of the binding no-arbitrage condition. In Section 4, we explain the estimation methods adopted in the paper. Section 5 presents the empirical results and the corresponding diagnostic statistics and Section 6 concludes.

2. Review of the Literature

Stochastic volatility (SV) models have been used extensively in the empirical finance literature; see Ghysels et al. (1996) for a comprehensive review. Many generalizations have been introduced in order to replicate specific aspects of the data and to improve the performance of continuous time models. These include:

i) *Stochastic volatility* in order to account for negative skewness and high kurtosis in the stock return series (see, e.g., Cox (1975), Cox (1996), Hull & White (1987), Scott (1987), Wiggins (1987), Stein & Stein (1991), Heston (1993), Naik (1993), Duan (1995) , Fouque et al. (2000), Davydov & Linetsky (2001), Detemple & Tian (2002));

ii) *Stochastic volatility and jumps in the stock price process* so as to improve the pricing of short term options (see, e.g., Bates (1996), Scott (1997), Carr et al. (2003));

iii) *Stochastic volatility and jumps in both the stock price and volatility processes* in order to achieve the empirically observed persistence in the impact of jumps (Duffie et al. (2000)).

If we consider the implied volatility (IV) derived from option prices as a proxy for the behavior of volatility for stock returns, then a major issue remains unexplored, as none of the

existing models is capable of accounting for *spikes* in the observed implied volatility. More specifically, in Duffie et al. (2000), volatility is modeled as an affine process that can jump up violently, but that subsequently cannot jump down as observed in the data. Therefore, as Eraker et al. (2003) observes, by relying on the assumption of *positive jumps* in the volatility process, the Duffie model can explain the abrupt increase in volatility registered on the day of the crash, but not the subsequent violent fall.

In the commodity pricing literature, spikes in the spot price process have been modeled and tested for some time (see Geman & Roncoroni (2006) for a review) while the volatility of the process is treated as constant. Our model differs from this literature not only in the way spikes are allowed, with no restrictions on the size of the jumps, as well as the way they are allowed to affect both the stock return process and the volatility process.

Empirical tests for jumps have been considered by Kobayashi (2005) who proposes a test for jumps in volatility for the case where jumps in returns and volatility are contemporaneous and correlated and jumps in volatility can be only positive, with the jump size exponentially distributed. In contrast, Lee & Mykland (2008) propose a non-parametric test to detect jump arrival times and realized jump sizes in asset prices. In the related literature on high frequency data some advances in SV models with jumps in volatility have been recently proposed: see in particular Todorov (2011) where volatility is a moving average of past jumps.

In this paper we test for spikes in volatility indirectly by arguing that in periods of high financial instability, evidenced by high kurtosis, taking account of spikes in volatility dramatically improves the empirical performance of the model.

As a framework for the volatility process, we assume that the logarithm of the variance follows an Ornstein–Uhlenbeck process.[3] This choice reflects the ability of the exponential function to generate high volatility values in a very limited time. Although many general-

[3]In discrete time, the counterpart of this model can be found in the EGARCH model of Nelson (1991). Alternatively, Scott (1987) assumes that the logarithm of volatility (the square root of the variance process) follows an Ornstein–Uhlenbeck process. Other branches of the literature model volatility as an Ornstein–Uhlenbeck process where the underlying state variable is Gaussian (Wiggins (1987)); Chesney & Scott (1989); Melino & Turnbull (1990)), as a CIR (Cox et al. (1985)) process with a reflecting barrier at zero where the underlying state variable is Gamma distributed (Cox et al. (1985)); Bailey & Stulz (1989); Heston (1993)) or as a CEV (constant elasticity of variance) process (Cox (1975); Cox & Ross (1976); Jones (2003)).

izations of the loglinear model have been introduced, there has been no consideration of the possibility of Poisson jumps within this framework.[4] The closest contribution in this direction is the model proposed by Duffie et al. (2000) who introduce jumps in a Cox, Ingersoll and Ross (CIR) model for volatility (affine process). Compared to this specification, the model proposed here is more flexible as in this framework:

- there are no constraints on the sign of the jumps in the volatility process;
- the stock price is able to jump both alone and together with volatility;
- the exponential function is suitable for modelling periods of market stress due to its ability to generate extremely high volatility values.

All these features enable our model to price a stock (or the market index) and its underlying derivatives even in the context of severe financial market stress.

3. The Model

The $\sigma-$field $\mathcal{F}_t$ represents the information that investors have at each point in time $t \in [0,1]$ with $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \leq t$. Suppose that $(\Omega, P, \mathcal{F})$ is the probability space for this model. More specifically, P is the probability measure which represents investors' beliefs, Ω is the set of states of the world and $\mathcal{F} \equiv \mathcal{F}_1$ is the set of events that can be seen at the trading horizon. The filtration $\mathbf{F} \equiv \{\mathcal{F}_t; t \in [0,1]\}$ is assumed to be right continuous and P-complete. In this economy there are N stocks. The price of the stock portfolio S_t, at time t is assumed to follow a mixture of Brownian motion and a multivariate compensated Poisson process. More specifically, the stock price process is right continuous (securities are traded ex-dividend) and left limited.[5]

Besides the bond $B_t = e^{-rt}$, there is a portfolio of risky assets and a stochastic volatility component.

The intuition behind these two equations is very simple. The stock price S_t is allowed to vary not only over time, but also in response to two kinds of shocks:

[4] Alizadeh et al. (2002), Chacko & Viceira (2003), and Gallant & Tauchen (1999) consider volatility as a multifactor diffusion model.

[5] One can assume, without loss of generality, that the left limit exists and is finite: $S_{nt^-} = \lim_{u \to t} S_{nu}$. Consequently, the jump of the stock price at time t will be: $\Delta S_{nt} = S_{nt} - S_{nt^-}$.

i) *diffusive shocks*, such as W_{1t}, W_{2t}, which affect the stock price gradually and by small amounts. Any diffusive shock affecting the volatility process impacts the stock price process through the "weak leverage effect" β_{12}.

ii) *Poisson driven shocks*, represented by the random measures P_1 and P_2, which account for abrupt and huge changes in the stock price. P_1 represents the number of jumps, with stochastic size ζ_1, experienced by the stock return over the time interval $(0, t]$. P_2 plays the same role as P_1, but it regulates the time varying impact exerted on the stock price by any shock to the corresponding volatility. In this case, the jump size is ζ_2. We are thus in a position to replicate abnormal market movements that take place when volatility is affected by huge shocks which are transmitted to the stock price through the "strong leverage effect" ψ_{33}.

$$\begin{aligned} d\ln S_t = \mu dt + \sqrt{V_t}\left[\beta_{12} dW_{2t} + \sqrt{1-\beta_{12}^2} dW_{1t}\right] \\ + \sqrt{1-\psi_{33}^2}\int_{R\setminus\{0\}} \zeta_1(\Gamma) P_1(d\Gamma, dt) + \psi_{33}\int_{R\setminus\{0\}} \zeta_2(\Gamma) P_2(d\Gamma, dt) \end{aligned} \tag{1}$$

where the stochastic part of the corresponding volatility follows the law

$$V_t = \exp(U_t)$$

$$dU_t = \left(\mu^U + \alpha_{22} U_t\right) dt + \beta_{20} dW_{2t} + \int_{R\setminus\{0\}} \zeta_3(\Gamma) P_2(d\Gamma, dt) \tag{2}$$

$$\begin{aligned} \zeta_1(\Gamma) &\sim N(\psi_{11}, \psi_{12}^2) \\ \zeta_2(\Gamma) &\sim N(\psi_{13}, \psi_{23}^2) \\ \zeta_3(\Gamma) &\sim N(\psi_{21}, \psi_{22}^2) \end{aligned}$$

where $\int_{R\setminus\{0\}} \zeta_i(\Gamma) P_j(d\Gamma, dt) - \mu_{\zeta_i}\lambda_j dt$ for $i = 1, 2, 3$ and $j = 1, 2$ are the compensated Poisson random measures. $\int_{R\setminus\{0\}} \zeta_i(\Gamma) P_j(d\Gamma, dt)$ counts the number of jumps with random size $\zeta_i(\Gamma)$ in the set $R\setminus\{0\}$ during the small time interval dt. $P_j(d\Gamma, dt) = 1$ only whenever the jump event of size $\zeta_i(\Gamma)$ occurs; $P_j(d\Gamma, dt) = 0$ in all other cases.

4. Estimation Method

The model of this paper could be estimated using maximum likelihood estimation (MLE), Semi-Non-Parametric (SNP) methods or the Simulated Method of Moments (SMM). However, as volatility is a latent variable, MLE would be too demanding and intensive from the computational point of view. Indeed, volatility should be integrated out of the likelihood function and the dimension of this integral would be as large as the number of observations in the time series. For similar reasons, SNP procedures are also not easily implemented. SMM, which uses Monte Carlo simulation to construct the moment conditions for the Generalized Method of Moments (GMM), is not efficient.

A possible alternative to this estimation method is to use the Efficient Method of Moments (EMM) to simultaneously estimate the parameters of the stock return process and the risk premia. However, while this technique appears more intuitive for its ability to directly provide estimates of the risk premia, it is vulnerable to the critique that the estimation based on a multidimensional auxiliary model suffers from poor finite sample properties (see Duffee & Stanton (2004)). Moreover, the use of return and option data at the same time is computationally so intensive that its implementation typically involves the use of very short data sets. An example can be found in Chernov & Ghysels (2000) who estimate the Heston model by EMM using both stock and option data. The multidimensional approach is also employed by Pan (2002) who estimates a square root model with jumps by GMM, by Eraker (2002) who evaluates the Duffie et al. (2000) model by the Markov Chain Monte Carlo (MCMC) technique and Jones (2003) who estimates a CEV (Constant Elasticity of Variance) model. Alternatively, the unidimensional approach, where only the stock return series is used, is chosen by Andersen et al. (2002) and Chernov et al. (2003) who use EMM to estimate the parameters of many possible models for the stock return process.

4.1. Estimating the stock return and volatility parameters by EMM

The parameters of the stock price and volatility processes are estimated via EMM. Specifically, the vector of parameters to be estimated at this point is (see equations [1] and [2]):

$$\xi_1 = \ (\alpha_{10}, \beta_{12}, \psi_{33}, \psi_{11}, \psi_{12}, \psi_{13}, \psi_{23}, \psi_{13}, \alpha_{20}, \alpha_{22}, \beta_{20}, \psi_{21}, \psi_{22}, \lambda_1, \lambda_2)$$

The initial step consists in choosing an appropriate transition density called the *auxiliary model* which is a close approximation of the data generating process. The parameters of this density are estimated by QMLE (Quasi Maximum Likelihood Estimation). The score of this model represents the score generator for EMM.

Following Gallant & Tauchen (2002), the auxiliary model is derived by using the so-called SNP (Semi-Non-Parametric) approach. This approach consists of approximating the unknown density of the stock returns using a Hermite expansion with a Gaussian leading term. The higher order terms of this expansion will accommodate any deviation from Gaussianity such as, for example, high kurtosis and negative skewness.

Let $f_K\left(y_t|\, x_{t-1}, a_n\right)$ denote the resulting density of the stock returns with a the vector of unknown parameters. The parameter vector a will be estimated by the QMLE method. Hence, $\widehat{a}_{QMLE}$ will be such that

$$\frac{1}{n}\sum_{t=0}^{n}\frac{\partial}{\partial a}\ln f_K\left(y_t|\, x_{t-1}, \widehat{a}_n\right) = 0$$

Let $s_f\left(Y_t, \widehat{a}_n\right) \equiv \frac{\partial}{\partial a_n}\ln f_K\left(y_t|\, x_{t-1}, \widehat{a}_n\right)$ be the score function and let

$$\widehat{Y}_T\left(\xi_1\right) = \left\{\widehat{y}_1\left(\xi_1\right), \widehat{y}_2\left(\xi_1\right), ..., \widehat{y}_T\left(\xi_1\right)\right\}$$

denote a simulated time series using the structural model with fixed parameter vector ξ_1.

Evaluating the score functions at this simulated series of data and keeping the parameters of the auxiliary model fixed at $\widehat{a}_{QMLE}$, we obtain a set of moment conditions

$$m_T\left(\xi_1, \widehat{a}_n\right) \equiv \frac{1}{T}\sum_{t=1}^{T} s_f\left(\widehat{Y}_t\left(\xi_1\right), \widehat{a}_n\right)$$

where

$$s_f\left(\widehat{Y}_t\left(\xi_1\right), \widehat{a}_n\right) \equiv \frac{\partial}{\partial a_n}\ln f_K\left(\widehat{y}_t|\, \widehat{x}_{t-1}\left(\xi_1\right), \widehat{a}_n\right)$$

The EMM estimator $\widehat{\xi}_{1n}$will be such that

$$\widehat{\xi}_{1n} = \arg\min\left\{m_T\left(\xi_1, \widehat{a}_n\right)\widehat{V}_n^{-1} m_T\left(\xi_1, \widehat{a}_n\right)'\right\}$$

where, following Gallant & Long (1997), a consistent estimator of the asymptotic covariance matrix of the sample score vector may be obtained by:

$$\widehat{V}_n = \frac{1}{n}\sum_{t=1}^{n} s_f\left(Y_t, \widehat{a}_n\right) s_f\left(Y_t, \widehat{a}_n\right)'.$$

5. Empirical Results

In this section, we describe the data, explain the EMM estimation and estimate seven other well-known models for stock pricing as benchmarks for our model.

5.1. Analysis of S&P 500 stock returns' statistical properties

We consider daily percentage index returns:

$$y_t = 100[\ln(S_t) - \ln(S_{t-1})]$$

on S_t, the S&P 500 stock index, for the period November 1973–December 2010: 9,380 business days. To evaluate the temporal stability of the stochastic process generating percentage returns, we compute the first four moments of the series with a moving-window procedure. The window width is set to approximately one year of trading data (252 business-daily observations). This fixed-width window is moved through the data by approximately three trading months (63 business days). The resulting dataset contains 216 observations, corresponding roughly to calendar quarters. The mean, standard deviation, skewness and kurtosis of these observations are presented in Figure 3.

5.2. Temporal stability

As our analysis is concerned with the volatility of the stock index returns series, we evaluate whether the relationship between the four moments exhibits temporal stability over the sample period. A breakpoint regression procedure of the type discussed by Hansen (2001) is performed for a regression model in which

$$\sigma_t = \beta_0 + \beta_1 \mu_t + \beta_2 sk_t + \beta_3 ku_t + \epsilon \tag{3}$$

where sk_t and ku_t refer to the moving-window estimates of skewness and kurtosis. This equation is augmented with an indicator variable for each time period included in the breakpoint period. The breakpoint period excludes the first 5% and last 5% of the observations. The indicator variable is interacted with each of the regressors, and the sum of squared errors (SSE) for each break period recorded. The minimum value of SSE is chosen as the break period.

Inspection of the sequence of minimum SSE values (see Figure 4) indicates that there are two points in time that may be considered breaks in the model's temporal stability: in

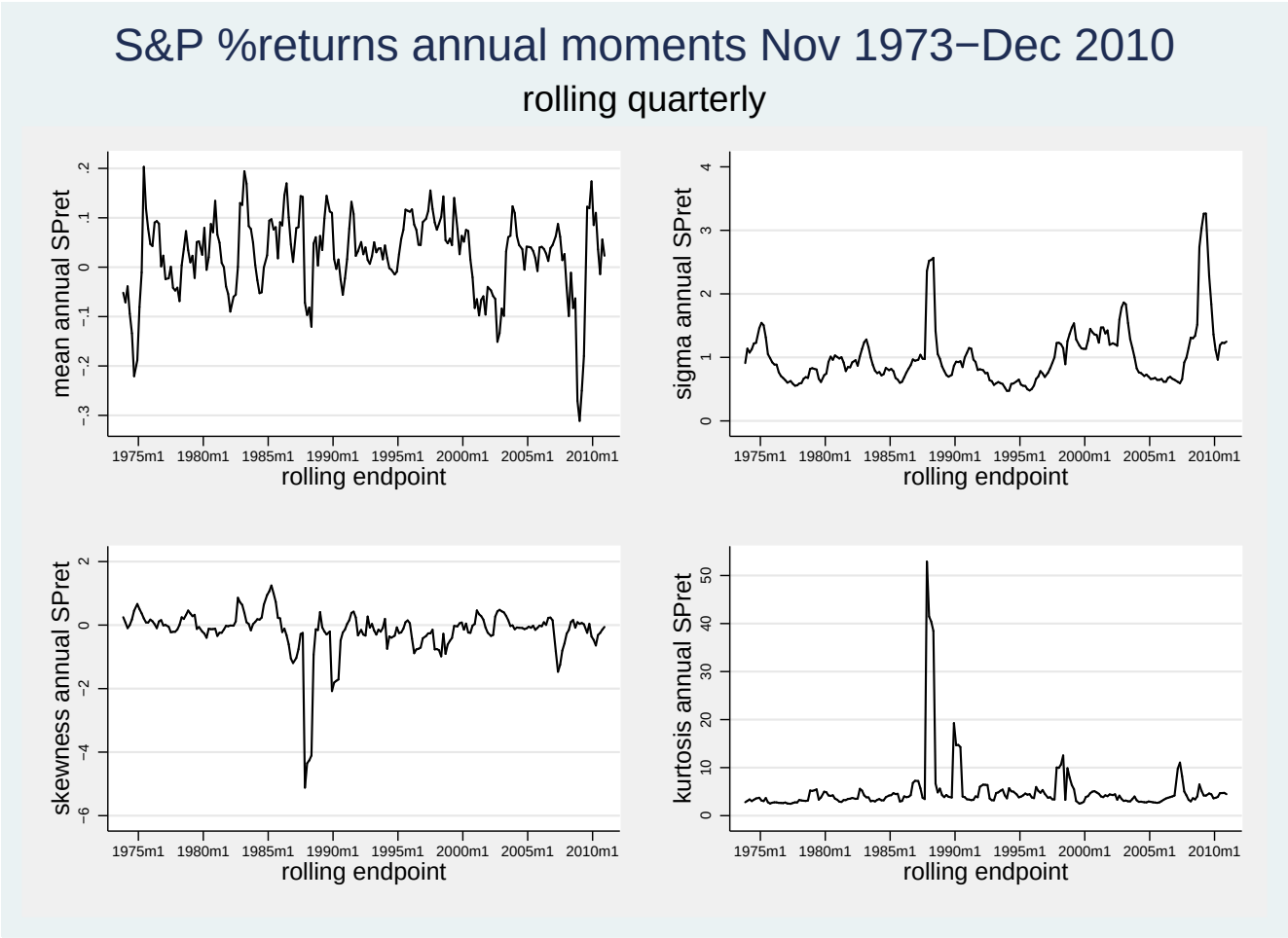

Figure 3: S&P 500 moments

June 1986 and July 1997, as indicated on the Figure. Consequently, further analysis of the S&P 500 index returns is performed separately for three periods.

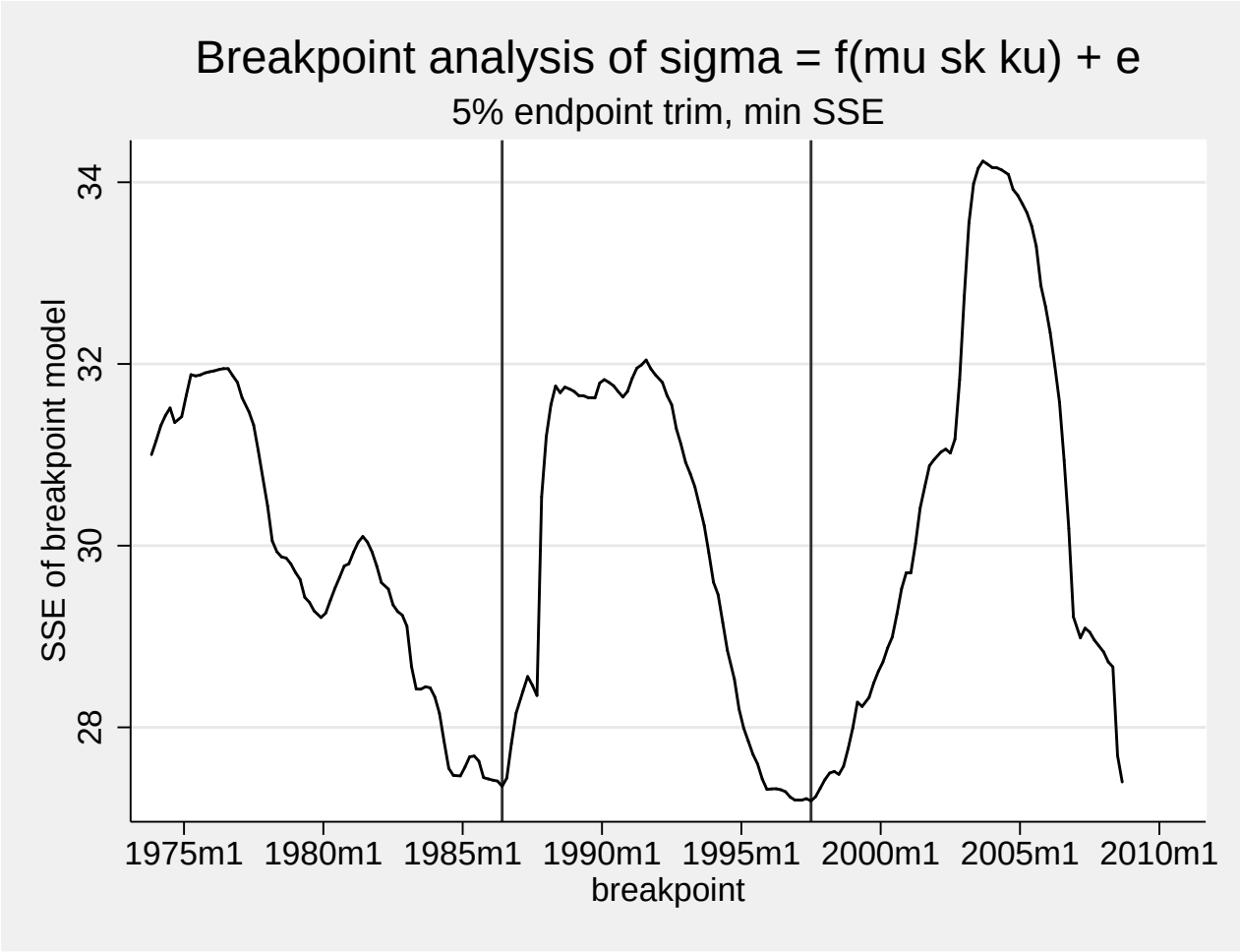

Figure 4: S&P 500 returns: analysis of temporal stability

The moments of the returns for each subperiod and for the entire sample are reported in Table 1. As is quite evident, the second subperiod, July 1986–July 1997, is quite unusual in terms of its mean, skewness and kurtosis. Its mean is twice that of the earlier period, and more than four times larger than its value in the third period. Likewise, the second subperiod is characterized by much larger negative skewness than the following period, while skewness in the first period is mildly positive. The kurtosis of the second subperiod is exceedingly large when contrasted with that of its predecessor or successor. Therefore, the data generating

Table 1: S&P returns: descriptive statistics by subperiod

	N	μ	σ	sk	ku
Nov 1973-Jun 1986	3178	0.026	0.907	0.216	4.598
Jul 1986-Jul 1997	2804	0.046	0.997	-4.695	109.098
Aug 1997-Dec 2010	3398	0.010	1.352	-0.164	10.128
Full Sample	9380	0.026	1.113	-1.088	29.651

process for stock index returns does not exhibit temporal stability, and a well-specified model of its properties must take that into account. Before we proceed to modeling the series, we evaluate the time series properties of the daily index returns series.

5.3. Time series properties of the returns series

For each of these three subperiods, we test the returns series and their squares for the presence of a unit root. Using the Elliott et al. (1996) DF-GLS test, each of these series produced rejections of the null hypothesis of a unit root at the 5% level of significance, and most rejected at the 1% level of significance.

For each regime, the returns series were modeled with an $AR(5)$ process, allowing for weekday effects in the autocorrelations of the returns process. Their residuals were analyzed for the presence of $ARCH(p)$ effects for $p = 1, 2, \ldots, 21$. The null hypothesis of no ARCH effects was decisively rejected for all values of p for each of the three residual series. GARCH(1,1) models were fit to the returns series for each subperiod using both a Gaussian and a t-distribution for the mean equations' errors. The results of these models are presented in Tables 2, 3, and 4 for the three subperiods.

In the first subperiod, Nov. 1973–June 1986, only the first lagged return is significant in the mean equation. The GARCH terms' values cannot reject the hypothesis of integrated GARCH (IGARCH). The degrees of freedom for the t-distributed errors, labeled lndfm2, is estimated at 14, reflecting the relatively low kurtosis of returns in this subperiod.

In the second subperiod, Jul. 1986–Jul. 1997, several lags in the mean equation are significant, and approximate IGARCH effects can be noted in the conditional variance equation. The degrees of freedom for t-distributed errors are estimated to be 4.6, reflecting the high kurtosis in this subsample.

In the third subperiod, Aug. 1997-Dec. 2010, several lags in the mean equation are again

Table 2: ARCH estimates S&P500 returns, Period 1: 11/1973–06/1986

	(1)	(2)
spret		
L.spret	0.154***	0.151***
	(8.31)	(8.19)
L2.spret	-0.0276	-0.0280
	(-1.43)	(-1.50)
L3.spret	-0.000118	-0.00115
	(-0.01)	(-0.06)
L4.spret	-0.0161	-0.0158
	(-0.87)	(-0.87)
L5.spret	-0.000268	0.000511
	(-0.02)	(0.03)
_cons	0.0253	0.0196
	(1.83)	(1.41)
ARCH		
L.arch	0.0435***	0.0405***
	(8.31)	(6.27)
L.garch	0.948***	0.952***
	(142.82)	(119.97)
_cons	0.00689**	0.00596**
	(3.29)	(2.59)
lndfm2		
_cons		14.45***
		(10.67)
N	3178	3178

t statistics in parentheses

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table 3: ARCH estimates S&P500 returns, Period 2: 07/1986–07/1997

	(1)	(2)
spret		
L.spret	0.0461*	0.0194
	(2.09)	(1.06)
L2.spret	-0.0173	-0.0335
	(-0.77)	(-1.83)
L3.spret	-0.0216	-0.0381*
	(-1.00)	(-2.13)
L4.spret	-0.0578**	-0.0296
	(-2.82)	(-1.66)
L5.spret	-0.0140	-0.0244
	(-0.69)	(-1.39)
_cons	0.0666***	0.0758***
	(4.47)	(5.97)
ARCH		
L.arch	0.0911***	0.0448***
	(28.74)	(5.95)
L.garch	0.894***	0.945***
	(150.25)	(116.63)
_cons	0.0173***	0.00833***
	(7.33)	(3.53)
lndfm2		
_cons		4.598***
		(6.49)
N	2804	2804

t statistics in parentheses

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table 4: ARCH estimates S&P500 returns, Period 3: 08/1997–12/2010

	(1)	(2)
spret		
L.spret	-0.0535**	-0.0560**
	(-2.64)	(-2.95)
L2.spret	-0.0217	-0.0436*
	(-1.27)	(-2.51)
L3.spret	-0.0207	-0.0251
	(-1.16)	(-1.44)
L4.spret	-0.0126	-0.0130
	(-0.73)	(-0.76)
L5.spret	-0.0455**	-0.0345*
	(-2.60)	(-2.02)
_cons	0.0545**	0.0702***
	(3.20)	(4.47)
ARCH		
L.arch	0.0842***	0.0775***
	(12.23)	(8.44)
L.garch	0.908***	0.919***
	(123.20)	(102.99)
_cons	0.0145***	0.00928**
	(6.69)	(3.04)
lndfm2		
_cons		7.747***
		(11.06)
N	3398	3398

t statistics in parentheses

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

significant, with near-IGARCH estimates in both specifications. The degrees of freedom for t-distributed errors are estimated to be 7.7, reflecting moderate kurtosis in this subsample.

5.4. Estimation of the stock return parameters by EMM

Let

$$y_t = 100[\ln(S_t) - \ln(S_{t-1})]$$

be the daily S&P stock index return for our estimation problem. $\{y_1, y_2, ..., y_n\}$ is then the available data set. We characterize as

$$x_{t-1} \equiv \{y_{t-L}, y_{t-L+1}, ..., y_{t-1}\}$$

the L lagged values of the realization of the time series $\{y_t\}_{t=-\infty}^{\infty}$.

Denote by H the finite dimensional Euclidean space where the likelihood functional is characterized. The likelihood can therefore be written as

$$\left[\prod_{t=1}^{n} p\left(y_t \middle| x_{t-1}, \xi_1\right)\right] \int p\left(y, x_0, \xi_{10}\right) dy$$

where

$$p\left(y_t \middle| x_{t-1}, \xi_1\right) = \frac{p\left(y_t, x_{t-1}, \xi_1\right)}{\int p\left(y, x_{t-1}, \xi_1\right) dy}$$

and ξ_1 is the parameter vector of this model.

Following Gallant & Tauchen (2002), by expanding $\left[p\left(y, x_{t-1}\right)\right]^{\frac{1}{2}}$ in an Hermite series and deriving the transition density of the truncated expansion, it is possible to calculate the transition density $f_K\left(y_t \middle| x_{t-1}\right)$ where

$$y_t = Rz_t + \mu_{x_{t-1}}$$

and R is an upper triangular matrix

$$vech(R_{x_{t-1}}) = \rho_0 + \sum_{i=1}^{L_r} P_i \left|y_{t-1-L_r+i} - \mu_{x_{t-2-L_r+i}}\right| + \sum_{i=1}^{L_g} diag(G_i) vech(R_{x_{t-2-L_g+i}})$$

and

$$\mu_x = b_0 + Bx_{t-1}$$

is the location function where b_0 is a vector and B is a matrix. The resulting standardized residual will be

$$z_t = R^{-1}\left(y_t - b_0 - Bx_{t-1}\right)$$

with corresponding density function

$$h_K\left(z_t \middle| x_{t-1}\right) = \frac{\left[P\left(z_t, x_{t-1}\right)\right]^2}{\int \left[P\left(u, x_{t-1}\right)\right]^2 \phi\left(u\right) du}$$

where $P\left(z_t, x_{t-1}\right)$ is the Hermite polynomial with rectangular expansion

$$P\left(z_t, x_{t-1}\right) = \sum_{j=0}^{K_z} \sum_{i=0}^{K_x} a_{ij} \left(x_{t-1}\right)^i z_t^j$$

where $a_0 = 1$ in order to achieve identification. $P\left(z_t, x_{t-1}\right)$ is a polynomial in z of degree K_z whose coefficients are polynomials of degree K_x. K_z is the order of the polynomial expansion that allows for deviations of the tails of the distribution from the Normal density.

$\phi\left(t\right)$is the standard normal density and the normalization term

$$\int \left[P\left(u, x_{t-1}\right)\right]^2 \phi\left(u\right) du$$

is such that the SNP density integrates to one. Using this SNP model, it is possible to derive the conditional density of y_t as

$$f_K\left(y_t \middle| x_{t-1}\right) = \frac{h_K\left[R_x^{-1}\left(y_t - \mu_x\right) \middle| x_{t-1}\right]}{\det\left(R_x\right)}$$

The Hermite expansion consists of a polynomial in z (which represents the innovation) multiplied by the standard Gaussian density. The flexibility of this model is the main reason why this might be considered as the best choice in order to approximate the data generating process. In fact, if $K_z = 0$, then this density function is just a standard Gaussian density. Any deviation from this can be taken care of just by allowing for $K_z > 0$.

When the coefficients a_i are not considered as functions of x_{t-1}but as constants, the density function of the innovation will be

$$h_K(z_t) = \frac{\left[\sum_{i=0}^{K_z} a_i z_t^i\right]^2 \phi(z_t)}{\int \left[\sum_{i=0}^{K_z} a_i u^i\right]^2 \phi(u)\, du}$$

this density will generate a Gaussian VAR if $K_z = 0$, while any departure from Gaussianity are accommodated by setting $K_z > 0$.

In order to model important aspects of the data such as the presence of conditional heteroskedasticity, it will be useful to assume that these coefficients a_i are actually functions of x_{t-1}:

$$a_i(x_{t-1}) = \sum_{j=0}^{K_x} a_{ij} x_{t-1}^j$$

This further generalization introduces a nonlinear conditional shape variation with x_{t-1}.

The conditional density of the innovations in this case will be

$$h_K(z_t) = \frac{\left[\sum_{i=0}^{K_z} \left(\sum_{j=0}^{K_x} a_{ij} x_{t-1}^j\right) z_t^i\right]^2 \phi(z_t)}{\int \left[\sum_{i=0}^{K_z} \left(\sum_{j=0}^{K_x} a_{ij} x_{t-1}^j\right) u^i\right]^2 \phi(u)\, du}$$

The only restriction, in this case, is that the dimension of a (the parameter vector of the auxiliary model) is greater or equal to the dimension of the parameter vector of the structural model ξ_1.

As already stated, we choose an auxiliary model whose main parameters include:

L_u	number of lags in the location function μ_x
L_g	number of lags in $vech(R_{x_{t-1}}, g, r)$ lags in GARCH (autoregressive)
L_r	number of lags in $vech(R_{x_{t-1}}, g, r)$ lags in ARCH (moving average)
L_p	number of lags in the x_{t-1} part of the polynomial $P(z_t, x_{t-1})$
K_z	degree of the polynomial $P(z_t, x_{t-1})$ in z_t
K_x	degree of the polynomial $P(z_t, x_{t-1})$ in x_{t-1}

Following Chernov et al. (2003), we choose the values of these parameters that minimize the BIC (Schwarz or Bayes information criterion). The final nonlinear, non-parametric auxiliary models we select for the three data sets are characterized by the parameters presented in Table 5.

The three auxiliary models all constitute GARCH(1,1) processes with different polynomial degrees in the Hermite expansion of the non-parametric error density function. For

Table 5: Parameters for the auxiliary model by subperiod

	11/1973–6/1986	7/1986–7/1997	8/1997–12/2010
Polynomial			
L_p	1	1	1
K_z	6	6	7
I_z	1	1	1
K_x	1	2	1
I_x	1	1	1
Mean function			
lag_μ	1	1	1
Variance function			
lag_g	1	1	1
lag_r	1	1	1
lag_p	19	25	20
sn	1.340646	1.17327	1.23124342
BIC	1.36496	1.20901294	1.25536513

Note: lag_μ specifies the lag order of the location function μ_x. lag_g and lag_r specify the order of the GARCH(g,r) model. lag_p specifies the lag order of the x_{t-1} polynomial.

the first and second subsets, we select a sixth-degree polynomial expansion (K_z) while for the third subset we select a seventh-degree Hermite polynomial expansion. Intuitively, when $K_z > 0$, the shape of the original Gaussian density is multiplied by a polynomial in z. For the second subset, the data indicate $K_x = 2$, which accounts for an higher conditional heterogeneity.

The restrictions implied by these specific choices of the auxiliary model's tuning parameters specify a general nonlinear process with heterogeneous innovations. The parameters K_z and K_x determine the degree of the polynomial $P(z_t, x_{t-1})$ and hence the nature of the innovation process $\{z_t\}$.

5.5. Benchmarking the model against established models

To evaluate the validity of the model proposed above, we must consider how well it satisfies the moment restrictions identified in the EMM procedure, and compare its performance with that of competing models in the established literature. The models we consider include:

Black–Scholes based models, which we designate as BS models:

- BS: the original Black & Scholes (1973) model, which analytically predicts a flat implied volatility surface
- BSJ: the Black & Scholes (1973) model, augmented to allow jumps in the stock return process, but with constant volatility, as developed in Andersen et al. (2002)

BS model:

$$d\ln S_t = \alpha_{10}dt + \beta_{10}dW_{1t}$$

BSJ model:

$$d\ln S_t = \alpha_{10}dt + \beta_{10}dW_{1t} + \int_{R\setminus\{0\}} \zeta_1(\Gamma)\, P_1(d\Gamma, dt)$$

Models in the class of those considered by Duffie et al. (2000), which we designate as SV2 models:

- SV2: a model with stochastic volatility only
- SV2J: a stochastic volatility model with jumps in the stock return process
- SV2IJ: a stochastic volatility model with jumps in both the stock return and volatility processes. The jumps in volatility can only be positive, as they are exponentially distributed.

SV2 model:

$$\begin{aligned} d\ln S_t &= \alpha_{10}dt + \sqrt{V_{t^-}}dW_{1t} \\ dV_t &= \alpha_{20}(\alpha_{22} - V_{t^-})\,dt + \sqrt{V_{t^-}}\beta_{20}\left[\beta_{12}dW_{1t} + \sqrt{1-\beta_{12}^2}dW_{2t}\right] \end{aligned}$$

SV2J model:

$$\begin{aligned} d\ln S_t &= \alpha_{10}dt + \sqrt{V_{t^-}}dW_{1t} + dJ^y \\ dV_t &= \alpha_{20}(\alpha_{22} - V_{t^-})\,dt + \sqrt{V_{t^-}}\beta_{20}\left[\beta_{12}dW_{1t} + \sqrt{1-\beta_{12}^2}dW_{2t}\right] \end{aligned}$$

SV2IJ model:

$$\begin{aligned} d\ln S_t &= \alpha_{10}dt + \sqrt{V_{t^-}}dW_{1t} + dJ^y \\ dV_t &= \alpha_{20}(\alpha_{22} - V_{t^-})\,dt + \sqrt{V_{t^-}}\beta_{20}\left[\beta_{12}dW_{1t} + \sqrt{1-\beta_{12}^2}dW_{2t}\right] + dJ^v \end{aligned}$$

Models which are generalizations of Nelson's loglinear model, in which the logarithm of the volatility process is modelled:

- SV1: a stochastic volatility model for log volatility (square root of the variance), based on Scott (1987)
- SV1J: a stochastic volatility (log-variance) model with jumps in the stock return process, based on Andersen et al. (2002)
- SV1CIJ: a stochastic volatility (log-variance) model with jumps in both the stock return and volatility processes, as developed above

SV1 model:

$$d\ln S_t = \alpha_{10}dt + \sqrt{V_t}\left[\beta_{12}dW_{2t} + \sqrt{1-\beta_{12}^2}dW_{1t}\right]$$
$$V_t = \exp(U_t)\ dU_t = (\alpha_{20} + \alpha_{22}U_t)\,dt + \beta_{20}dW_{2t}$$

SV1J model:

$$d\ln S_t = \alpha_{10}dt + \sqrt{V_t}\left[\beta_{12}dW_{2t} + \sqrt{1-\beta_{12}^2}dW_{1t}\right] + \int_{R\setminus\{0\}} \zeta_1(\Gamma)\,P_1(d\Gamma, dt)$$
$$V_t = \exp(U_t)\ dU_t = (\alpha_{20} + \alpha_{22}U_t)\,dt + \beta_{20}dW_{2t}$$

SV1CIJ model:

$$d\ln S_t = \alpha_{10}dt + \sqrt{V_t}\left[\beta_{12}dW_{2t} + \sqrt{1-\beta_{12}^2}dW_{1t}\right] + \sqrt{1-\psi_{33}^2}\int_{R\setminus\{0\}} \zeta_1(\Gamma)\,P_1(d\Gamma, dt) + \psi_{33}\int_{R\setminus\{0\}} \zeta_2(\Gamma)\,P_2(d\Gamma, dt)$$
$$V_t = \exp(U_t)\ dU_t = (\alpha_{20} + \alpha_{22}U_t)\,dt + \beta_{20}dW_{2t} + \int_{R\setminus\{0\}} \zeta_3(\Gamma)\,P_2(d\Gamma, dt)$$

$$\zeta_2 \sim N(\psi_{13}, \psi_{23}^2) \quad \zeta_1 \sim N(\psi_{11}, \psi_{12}^2) \quad P_1 \sim Poisson(\lambda_1) \quad P_2 \sim Poisson(\lambda_2)$$
$$\zeta_3 \sim N(\psi_{21}, \psi_{22}^2)$$

5.6. Estimates from the full sample period

We first present a set of estimates of our model and several alternative specifications for the entire sample period, November 1973–December 2010, in Table 6 for the SV2 models and Table 7 for the SV1 models. Both tables include the BS models as a benchmark. Focusing on the χ^2 statistics, which test satisfaction of the moment conditions, we find that all five SV2 models are rejected for the full sample, as are the three SV1 models reported in Table 7. However, the SV1CIJ model comes closest to satisfying the moment conditions.

Table 6: EMM estimates for the full sample November 1973–December 2010: BS and affine models

	SV2	SV2J	SV2IJ
α_{10}	0.0311527 [0.71294915]	0.035944233 [3.66221]	0.03023252
α_{20}	0.04360846 [6.005701]	0.016903059 [12.87673]	0.05212223
α_{22}	0.92773870 [12.467758]	1.0879803 [13.03783]	0.4275454
β_{12}	−0.28370491 [−3.875409]	−0.407813 [−4.718205]	−0.4303307
β_{10}			
β_{20}	0.168272106 [15.89719]	0.1195266 [16.78426]	0.077056757
ψ_{11}		−4.448700 [−30.75608]	−3.874262
ψ_{12}		0.4228922 [6.641593]	0.569167
ψ_{13}			1.92987
λ_{10}		0.0054566559847 [373.673233]	0.005528941
λ_{20}			0.0561991
χ^2 (df)	57.9562[10]	13.8813[7]	19.3517[5]
10% cv	15.987	12.017	9.236
5% cv	18.307	14.067	11.070

As another diagnostic of the models' performance, we may examine the quasi-t ratios for the semi-nonparametric moment conditions of the auxiliary model.

5.7. Estimates from Period I: November 1973–June 1986

Estimation results for the first subperiod, from November 1973 through June 1986, are presented in Tables 8 and 9 for SV2 and SV1 models, respectively. All of the BS and SV2 models are strongly rejected by the χ^2 statistics, as are the three SV1 models in Table 9.

Table 7: EMM estimates for the full sample November 1973–December 2010: BS and loglinear models

	BS	BSJ	SV1	SV1J	SV1CIJ
α_{10}	0.0562316 [7.66743]	0.091810 [2.967812]	0.0308592 [3.235180]	0.141864 [4.95176145]	0.0381711 [2.27939]
α_{20}			−0.455069 [−4.595626]	−0.5005016 [−7.78856]	−0.0074563 [−0.995711]
α_{22}			−1.163803 [−34.3359]	−0.046107 [−19.280644]	−0.03425905 [−1.35311]
β_{10}	0.75692 [76.5890]	0.450460 [14.73338]			
β_{12}			−0.1176739 [−4.9794]	−0.99146 [−148.10876]	0.069269146 [0.32332]
β_{20}			0.4668804 [25.59067]	1.524906 [12.84034]	0.11880691 [2.35766]
ψ_{11}		−0.0734479 [−1.74435]		−0.0253305 [−5.2156]	−3.013844 [−1.06104]
ψ_{12}		0.792822 [28.59171]		0.408390 [28.56021]	1.56459412 [1.294473]
ψ_{21}					0.29786937 [0.895110]
ψ_{22}					1.10529 [3.862170]
ψ_{13}					
ψ_{23}					0.23447892 [0.162018]
ψ_{33}					0.37596133 [0.189626]
λ_{10}		0.71358003 [117.07593]		0.1854397 [97.27136]	0.0070801672 [38.418351]
λ_{20}					0.00792787314 [0.900871]
χ^2 (df)	128.9063[13]	51.5983[10]	52.5286[10]	16.5680[7]	11.3930[2]
10% cv	19.812	15.987	15.987	12.017	4.605
5% cv	22.362	18.307	18.307	14.067	5.991

Table 8: EMM estimates for Period I: November 1973–June 1986: BS and affine models

	SV2	SV2J	SV2IJ
α_{10}	0.016193027 [1.114789]	0.0098101 [0.57175]	0.0430050 [2.765032]
α_{20}	0.01770211834 [1.5218787]	0.01546107 1.53057	1.43277 [3.2445]
α_{22}	0.52105140 [12.04433]	0.4883363 12.2820	0.43511 [13.99878]
β_{12}	−0.06066831147 [−0.22717]	0.03037337 [0.09365]	0.11427 [0.73084]
β_{10}	0.051689336 [2.53027]		
β_{20}		−0.0517339 [−2.66233]	0.263404 [2.77703]
ψ_{11}		1.34219 [0.992004]	0.647340 [0.27539]
ψ_{12}		−0.372609 [−0.24305]	1.23583 [0.42561]
ψ_{13}			0.25211731 [0.188583]
λ_{10}		0.001067498 [1.3047073]	0.00306355361 [0.54925]
λ_{20}			0.0048672999 [0.17789]
χ^2 (df)	83.4048[11]	80.2141[8]	74.826[6]
10% cv	17.275	13.362	10.645
5% cv	19.675	15.507	12.592

Table 9: EMM estimates for Period I: November 1973–June 1986: BS and loglinear models

	BS	BSJ	SV1	SV1J	SV1CIJ
α_{10}	0.0197076 [1.59127]	0.0119459 [0.70642]	0.0207120236 [0.207120236]	0.0398745 [2.3247365]	0.135686599 [0.43736082]
α_{20}			−0.0103367704 [−1.4400]	−5.020958 [−0.747513]	−1.797768453 [−2.76984]
α_{22}			−0.01451553 [−1.50038]	−5.292705 [−0.77797]	−1.9531735 [−9.96696]
β_{10}	0.6584834 [43.4959]	0.61711 [34.30657]			
β_{12}			0.14655855 [0.14655855]	−0.31034601 [−2.05237]	0.0359059804 [0.08559]
β_{20}			0.0693309362 [0.693309362]	−1.545346 [−1.182849]	−0.138323627 [−0.313635]
ψ_{11}		0.237049 [1.50062]		−0.2903712 [−0.040580]	−0.170299582 [−0.27368]
ψ_{12}		0.828243464 [6.5700]		3.750121 [0.225304]	−0.05435315254 [−0.068554]
ψ_{21}					0.326450765 [0.0380740]
ψ_{22}					0.05247372592 [0.0608103]
ψ_{13}					
ψ_{23}					564.40325 [0.18389]
ψ_{33}					0.000109389560 [0.00001]
λ_{10}		0.08718664 [12.84178]		0.000043660265 [3.650576]	0.51342334 [6.337547]
λ_{20}					0.0080338146 [0.20480]
χ^2 (df)	97.4580[14]	88.8337[11]	82.0387[11]	64.8826[8]	68.0459[3]
10% cv	21.064	17.275	17.275	13.362	6.251
5% cv	23.685	19.675	19.675	15.507	7.815

Thus, it appears that all models tested have weaknesses in modeling the behavior of stock returns and their volatility over this subperiod.

5.8. Estimates from Period II: July 1986–July 1997

Table 10: EMM estimates for Period II: July 1986–July 1997: BS and affine models

	SV2	SV2J	SV2IJ
α_{10}	0.0799066 [5.6194613]	0.09665205 [4.92614]	0.075335 [3.24266]
α_{20}	0.4133988 [2.36865]	1.770400 [6.805131]	1.898026 [10.1222]
α_{22}	0.4165384 [7.49780]	0.451205832 [4.44117]	0.476482 [3.4445]
β_{12}	0.374536 [1.92056]	0.768549395 [1.21225]	0.23999986 [0.28434]
β_{10}			
β_{20}	0.281851 [5.87722]	0.18214046 [1.51953]	0.1316202 [0.89240]
ψ_{11}		-1.3886887 [-2.86522]	-0.397528936 [-1.2776]
ψ_{12}		0.86905678999 [1.66107]	1.325589 [3.80379]
ψ_{13}			-3.846667 [-0.17540]
λ_{10}		0.026222736 [6.35045]	0.03101838 [461.538]
λ_{20}			0.005183211 [0.1020]
χ^2 (df)	39.3632[10]	35.5344[7]	20.8614[5]
10% cv	15.987	12.017	9.236
5% cv	18.307	14.067	11.070

Estimation results for the second subperiod, from July 1986 through July 1997, are presented in Tables 10 and 11 for SV2 and SV1 models, respectively. This subperiod encompasses the October 1987 stockmarket crash. In Table 10 and Table 11, we may see that only our model, the SV1 model with jumps in both returns and volatility, is not rejected by the data in terms of its χ^2 value. Interestingly enough, our log-volatility model performs much better in a subperiod containing extreme movements in the stockmarket index than it does in the earlier, generally more tranquil period.

Table 11: EMM estimates for Period II: July 1986–July 1997: BS and loglinear models

	BS	BSJ	SV1	SV1J	SV1CIJ
α_{10}	0.0772210 [5.597068]	0.09306499 [3.6961]	0.07190144 [4.28949]	0.1016 [2.3422]	0.1423908 [4.01896123]
α_{20}			−1.250 [−3.973]	−0.0030054 [−0.4292]	−0.040530203 [−0.800335]
α_{22}			−1.4924 [−8.515]	−0.0016397 [−0.4759]	−0.022429648 [−1.16666]
β_{10}	0.648986717 [14.20358]	0.3793854 [8.40397]			
β_{12}			0.88014 [9.13648]	−0.270219 [−0.18974]	0.79392 [3.24275]
β_{20}			0.5179150 [5.454914]	0.067950311 [0.75087]	0.186851 [0.68784]
ψ_{11}		−0.05168689 [−1.11539]		−0.0324931 [−0.915861]	−0.100774774 [−1.07538]
ψ_{12}		0.690576 [15.31877]		0.510799 [11.0900]	0.63050774 [1.3272]
ψ_{21}					−17.3671 [−3.3778]
ψ_{22}					13.37604 [4.98118]
ψ_{13}					1.3965 [0.001380]
ψ_{23}					27.542836 [0.001376]
ψ_{33}					0.0312346 [0.0013752]
λ_{10}		0.5395232 [55.21441]		0.0488781 [13.80290]	0.54046483 [21.81985]
λ_{20}					0.001204818 [0.45782]
χ^2 (df)	72.8018[13]	29.6264[10]	25.1529[10]	18.38[7]	1.2919[1]
10% cv	19.812	15.987	15.987	12.017	2.706
5% cv	22.362	18.307	18.307	14.067	3.841

Table 12: EMM estimates for Period III: August 1997–December 2010: BS and affine models

	SV2	SV2J	SV2IJ
α_{10}	−0.042366038 [−0.20397]	0.014802000 [0.710431]	0.01161460 0.47708
α_{20}	0.011082021 [4.959558]	0.00863020295 [8.15417]	0.0104512 2.59717
α_{22}	2.009403 [7.60872]	1.740817 [4.13879]	1.713096 2.4334
β_{12}	−0.71137 [−3.67336]	−0.938691 [−6.54188]	−0.99672138 −27.5472
β_{10}			
β_{20}	0.16977 [11.516311]	0.156252 [8.38224]	0.1802239 4.13577
ψ_{11}		−4.00178 [−2.0619]	−2.42318171 −4.19059
ψ_{12}		0.18310558 [0.026369]	0.02816455 0.007946
ψ_{13}			1.8826256 0.536587
λ_{10}		0.0049715342 [22.43955]	0.0086447946 2.5959297
λ_{20}			0.0050748386 1.167182
χ^2 (df)	33.7249[11]	25.1501[8]	20.9339[6]
10% cv	15.987	13.362	10.645
5% cv	18.307	15.507	12.592

Table 13: EMM estimates for Period III: August 1997–December 2010: BS and loglinear models

	BS	BSJ	SV1	SV1J	SV1CIJ
α_{10}	−0.053997 [−2.2852]	0.02268902 [0.84276]	−0.0306346 [−1.33050]	0.3872574 [1.1459011]	0.0430289 [0.52893]
α_{20}			0.827633151 [3.3590]	−0.00443321 [−0.3575575]	0.003173521 [0.051972]
α_{22}			−1.73459993 [−17.82101]	−0.00977295 [−1.944449]	−0.01175055 [−1.77850]
β_{10}	1.5100 [22.036964]	1.237297 [13.19031]			
β_{12}			−0.8127432 [−15.3453]	−0.89498735 [−19.08624]	−0.904873 [−9.75749]
β_{20}			0.306654540 [4.01378]	0.20927285 [1.85772]	0.1540752731 [1.09700]
ψ_{11}		−1.88452 [−2.64830]		−0.08930065128 [−1.16276]	−0.5946577 [−0.40879]
ψ_{12}		1.545987 [6.412399]		0.30249629 [2.35584]	1.1160117 [0.852266]
ψ_{21}					−0.4966358 [−0.0622]
ψ_{22}					0.48816915 [0.178629]
ψ_{13}					
ψ_{23}					−1.8163421457 [−0.011027]
ψ_{33}					0.09386244 [0.017582]
λ_{10}		0.03091839 [602.4559]		0.18360887 [83.14866]	0.072735 [7.79337]
λ_{20}					0.0067358317 [1.039822]
χ^2 (df)	56.754[14]	53.1713[11]	39.9586[11]	22.7373[8]	19.0491[3]
10% cv	21.064	15.987	15.987	13.362	6.251
5% cv	23.685	18.307	18.307	15.507	7.815

5.9. Estimates from Period III: August 1997–December 2010

Estimation results for the third subperiod, from August 1997 through December 2010, are presented in Tables 12 and 13 for SV2 and SV1 models, respectively. This subperiod encompasses the Great Recession and the ensuing stockmarket turbulence during the last several years. In Table 12, we see that all of the BS and SV2 models are rejected by the data on the basis of their χ^2 statistics. The SV1 models in Table 13 are also rejected by the data. In this period as well, it appears that all models tested have some significant flaws in capturing all aspects of the stock returns and volatility processes.

6. Conclusions

This ongoing work seeks to provide a model of stock returns' behavior and volatility with greater flexibility to handle periods of financial market stress. Although results are preliminary, it appears that modeling of these phenomena must take account of potential structural instabilities over a longer period. Analysis of three subperiods, identified by the data, indicates that the model developed in this paper performs quite well in the second subperiod, encompassing the extreme events surrounding the October 1987 stockmarket crash. During the earlier and later subperiods studied, including the more recent subperiod including the Great Recession, we do not find that any of the models tested provide a wholly adequate explanation of the dynamics of stock returns and their volatility.

References

Alizadeh, S., Brandt, M. W. & Diebold, F. (2002), 'Range based estimation of stochastic volatility models', *Journal of Finance* **57**(3), 1047–1091.

Andersen, T. G., Benzoni, L. & Lund, J. (2002), 'An empirical investigation of continuous-time equity return models', *Journal of Finance* **57**, 1239–84.

Bailey, W. & Stulz, R. (1989), 'The pricing of stock index options in a general equilibrium model', *Journal of Financial and Quantitative Analysis* **24**(1), 1–12.

Bates, D. S. (1996), 'Jumps and stochastic volatility: Exchange rate processes implicit in Deutsche Mark options', *Review of Financial Studies* **9**(1), 69–107.

Black, F. & Scholes, M. S. (1973), 'The pricing of options and corporate liabilities', *Journal of Political Economy* **81**(3), 637–54.

Carr, P., Geman, H., Madan, D. & Yor, M. (2003), 'Stochastic volatility for Levy processes', *Mathematical Finance* **13**(3), 345–382.

Chacko, G. & Viceira, L. M. (2003), 'Spectral GMM estimation of continuous-time processes', *Journal of Econometrics* **116**(1-2), 259–292.

Chernov, M., Gallant, A. R., Ghysels, E. & Tauchen, G. (2003), 'Alternative models for stock price dynamics', *Journal of Econometrics* **116**(1-2), 225–257.

Chernov, M. & Ghysels, E. (2000), 'A study towards a unified approach to the joint estimation of objective and risk neutral measures for the purpose of options valuation', *Journal of Financial Economics* **56**(3), 407–458.

Chesney, M. & Scott, L. (1989), 'Pricing European currency options: A comparison of the modified Black–Scholes model and a random variance model', *Journal of Financial and Quantitative Analysis* **24**(03), 267–284.

Cox, D. (1996), 'The constant elasticity of variance option pricing model', *Journal of Portfolio Management* **22**, 15–17.

Cox, J. (1975), Notes on options re: constant elasticity of variance diffusion, Working paper, Stanford University.

Cox, J. C. & Ross, S. A. (1976), 'The valuation of options for alternative stochastic processes', *Journal of Financial Economics* **3**(1-2), 145–166.

Cox, J., Ingersoll, J. & Ross, S. (1985), 'A theory of the term structure of interest rates', *Econometrica* **53**, 385–407.

Davydov, D. & Linetsky, V. (2001), 'The valuation and hedging of barrier and look back options under the CEV process', *Management Science* **47**, 949–965.

Detemple, J. & Tian, W. (2002), 'The valuation of American options for a class of diffusion processes', *Management Science* **48**, 917–937.

Duan, J.-C. (1995), 'The GARCH option pricing model', *Mathematical Finance* **5**(1), 13–32.

Duffee, G. R. & Stanton, R. H. (2004), Estimation of dynamic term structure models, Working paper, Haas School of Business U.C. Berkeley.

Duffie, D., Pan, J. & Singleton, K. (2000), 'Transform analysis and asset pricing for affine jump diffusions', *Econometrica* **68**, 1343–1376.

Elliott, G., Rothenberg, T. J. & Stock, J. H. (1996), 'Efficient tests for an autoregressive unit root', *Econometrica* **64**(4), 813–36.

Eraker, B. (2002), Do stock prices and volatility jump? Reconciling evidence from spot and option prices, Working paper, Graduate School of Business, University of Chicago.

Eraker, B., Johannes, M. & Polson, N. (2003), 'The impact of jumps in volatility and returns', *Journal of Finance* **58**(3), 1269–1300.

Fouque, J.-P., Papanicolaou, G. & Sircar, R. (2000), *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press.

Gallant, A. R. & Long, J. (1997), 'Estimating stochastic differential equations efficiently by minimum Chi-squared', *Biometrika* **84**, 125–141.

Gallant, A. R. & Tauchen, G. (1999), 'The relative efficiency of method of moments estimators', *Journal of Econometrics* **92**, 149–172.

Gallant, A. R. & Tauchen, G. (2002), Efficient method of moments, Working Paper 02–06, Department of Economics, University of North Carolina.

Geman, H. & Roncoroni, A. (2006), 'Understanding the fine structure of electricity prices', *Journal of Business* **79**(3), 1225–1262.

Ghysels, E., Harvey, A. & Renault, E. (1996), Stochastic volatility, *in* M. G.S. & C. Rao, eds, 'Handbook of Statistics: Statistical Methods in Finance', North-Holland, Amsterdam, pp. 119–191.

Hansen, B. E. (2001), 'The new econometrics of structural change: Dating breaks in U.S. labour productivity', *Journal of Economic Perspectives* **15**(4), 117–128.

Heston, S. L. (1993), 'A closed-form solution for options with stochastic volatility with applications to bond and currency options', *Review of Financial Studies* **6**(2), 327–43.

Hull, J. C. & White, A. D. (1987), 'The pricing of options on assets with stochastic volatilities', *Journal of Finance* **42**(2), 281–300.

Jones, C. S. (2003), 'The dynamics of stochastic volatility: Evidence from underlying and options markets', *Journal of Econometrics* **116**(1-2), 181–224.

Kobayashi, M. (2005), 'Testing for volatility jumps in the stochastic volatility process', *Asia-Pacific Financial Markets* **12**(2), 143–157.

Lee, S. S. & Mykland, P. A. (2008), 'Jumps in financial markets: A new nonparametric test and jump dynamics', *Review of Financial Studies* **21**(6), 2535–2563.

Melino, A. & Turnbull, S. M. (1990), 'Pricing foreign currency options with stochastic volatility', *Journal of Econometrics* **45**(1-2), 239–265.

Naik, V. (1993), 'Option valuation and hedging strategies with jumps in the volatility of asset returns', *Journal of Finance* **48**(5), 1969–84.

Nelson, D. B. (1991), 'Conditional heteroskedasticity in asset returns: A new approach', *Econometrica* **59**(2), 347–70.

Pan, J. (2002), 'The jump-risk premia implicit in options: Evidence from an integrated time-series study', *Journal of Financial Economics* **63**(1), 3–50.

Scott, L. (1997), 'Pricing stock options in a jump-diffusion model with stochastic volatility and stochastic interest rates: Application of Fourier inversion methods.', *Mathematical Finance* **7**, 413–426.

Scott, L. O. (1987), 'Option pricing when the variance changes randomly: Theory, estimation, and an application', *Journal of Financial and Quantitative Analysis* **22**(04), 419–438.

Stein, E. M. & Stein, J. C. (1991), 'Stock price distributions with stochastic volatility: An analytic approach', *Review of Financial Studies* **4**(4), 727–52.

Todorov, V. (2011), 'Econometric analysis of jump-driven stochastic volatility models', *Journal of Econometrics* **160**(1), 12–21.

Wiggins, J. B. (1987), 'Option values under stochastic volatility: Theory and empirical estimates', *Journal of Financial Economics* **19**(2), 351–372.