A Behavioral Model of Bubbles and Crashes

Taisei Kaizoji[1]

Abstract

The aim of this paper is to provide one potential theoretical explanation for questions how asset bubbles come about, why it persists, and what causes a crash. To do this, we propose a new model of bubbles and crashes. We consider an asset market in which the risky assets into two classes, the bubble asset and the non-bubble asset, and the risk-free asset are traded. Investors are divided into two groups of investors who have the different rationality of decision-making respectively. One is rational investors who maximize their expected utility of their wealth in the next period following their rational assessment of the fundamental values of risky assets. Another is noise traders who maximize their random utility of binary choice: holding the bubble asset and holding the risk-free asst. The noise trader's behavior is modeled in a framework of the theory of discrete choice with social interaction (Brock and Durlauf (1999, 2001)), which can be considered as a model of Keynse's beauty contest metaphor. We elucidate a mechanism that (i) noise-traders' herd behavior gives cause to a bubble, and (ii) their momentum trading prolongs bubble, (iii) a bubble is necessarily ended up with a crash, and (iv) the cycles of bubble and crash are repeated.

Key words: bubbles, crashes, noise traders, herd behavior, and Keynesian beauty contest.

JEL:

C73 - Stochastic and Dynamic Games; Evolutionary Games; Repeated Games

G01 - Financial crises

G17 - Financial Forecasting

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1. Introduction

In the last few decades, the asset markets have been frequently visited by bubbles and the subsequent crashes. The increasingly frequent market crashes have attracted the attention of the general public. Although many academics, practitioners and policy makers have studied questions related to collapsing asset price bubbles, the questions, how asset bubbles come about, why it persists, and what causes a crash, have been the greatest myths. What is the origin of bubbles? Why are asset prices deviated away from fundamental value?

It is widely believed that the internet stocks were in the midst of stock price bubble in the [1]period of the internet bubble from 1998 to 2000[2]. Brunnermeier and Nagel (2004) extracted hedge fund holdings from Form 13F, including those of well-known managers such as Soros, Tiger, Tudor, and others in the period of the internet bubble. They found that, over the sample period, many hedge fund managers tried to ride rather than attack bubbles, suggesting the existence of mechanisms that non-rational investors to surf bubbles rather than attempt to arbitrage them. Using manager age as a proxy for experience, Greenwood and Nagel (2008) study the portfolio decisions of experienced and inexperienced mutual fund managers during the internet bubble. They found that at the start of the bubble, the portfolios of younger managers, who have not yet directly experienced the consequences of a stock market downturn, show little deviation from those of older managers, but leading up to the peak in March 2000, younger managers strongly increase their holdings of technology stocks, while older managers do not. Brennan (2004) insists that increased stock market participation by individuals with little investment experience may have been the driving factor of the internet bubbles. Another recent growing body of empirical literature is also devoted to the existence of the momentum trading (also referred to as positive-feedback trading). Many empirical studies documents that the momentum in stock prices is positive in the short term, but eventually reversed in the long term. Among many literatures, the coexistence of the short-run momentum and the long-run reversal in stock prices is documented in detail by

[2] The internet bubble is investigated by Ofek and Richardson (2003), and Battalio and Schultz (2006).

Jegadeesh and Titman (1993), and DeBondt and Thaler (1985). Many researchers believe that the empirical evidences on the momentum trading (positive feedback trading) prove the existence of the noise-traders' herd behavior, and their herding have potential to explain speculative bubbles (see for example, DeLong et al. (1990)). These empirical findings are also consistent with results of experiments in laboratory asset markets. Smith, Suchanek, and Williams (1988) find that (a) bubbles and crashes occur regularly in laboratory asset markets when market participants are inexperienced, but (b) price gradually approach fundamentals when the participants, who have experienced bubbles and crashes in prior trading sessions, interact repeatedly in similar markets. Haruvy, Lahav, and Noussair (2009) finds that the investors' expectations of prices are adaptive, and primarily based on past trends in the previous and current laboratory asset markets in which they have participated. Most traders do not anticipate market downturns the first time they participate in a laboratory market, and are more prone to the optimism that fuels the bubble. In the opposite direction, when experienced, they typically exercise caution about market bubbles and crashes. In summary, the studies mentioned above indicate that the bubble is caused by the non-rational investors who attempt to surf bubble.

The recent theoretical literature on bubbles and crashes has evolved to increasingly recognize the evidence of bubbles which is defined as deviations from fundamental value. One important class of finance theories shows that there can be large movements in asset prices due to the combined effects of heterogeneous beliefs and short sales constraints. The basic idea finds its root back to the Lintner (1969)'s CAPM model of asset prices with investors having heterogeneous beliefs. (See, for example, Miller (1977), Jarrow (1980), Harrison and Kreps (1978), Chen, Hong and Stein (2000), Scheinkman and Xiong (2003), Hong and Stein (1999, 2003) and Hong, Scheinkman and Xiong (2006)). Another extensive body of literature is devoted to the concept of noise-trader (also referred to as positive-feedback investors) which is introduced first by Kyle (1985) and Black (1986) to describe irrational investors, and is developed first by De Long, Shleifer, Summers and Waldmann (1990a, 1990b).) Abreu and Brunnermeier (2003) propose a different mechanism justifying why rational traders ride rather than arbitrage bubbles

under the condition in which the stock price is kept above its fundamental value by irrationally exuberant behavioral traders such as noise-traders.

Independently of the recent studies, Keynes (1936 Chap. 12) proposes the completely different concept, that is, "beauty contest" to explain price fluctuations in stock markets. Keynes thought that similar behavior to his beauty contest metaphor was at work within the stock market. Investors evaluate shares not based on what they think their fundamental value is, but rather on what they think everyone else thinks their value is. More recently, the models of the stock market in terms of Keynes's beauty contest are proposed by Biais and Bossaerts, (1998), Allen, Morris, and Shin (2006), and Angeletos, Lorenzoni, and Pavan (2010).

In this paper we propose a new model to explore a mechanism of bubble and its subsequent collapse. We consider a stock market that the two contrary types of investors coexist. One is rational investors who invest based on their fundamental value they predict, whereas another is noise-traders whose investment is driven by expectations about what other investors predict, rather than expectations on their fundamentals. The rational investors are corresponded to experienced managers who have directly experienced the consequences of asset market crashes while noise traders are young managers who have not yet directly experienced the consequences of a stock market downturn. Our model shows that coexistence of the two contrary types of investors in the asset market is the key to understand a mechanism of stock market bubble and its subsequent crash. More concretely, we consider an asset market in which three assets: the bubble asset, the non-bubble asset, and the risk-free asset are traded. In accordance with traditional asst-pricing models (see e.g., Mossin (1966) and Lintner (1969)), the rational investors chooses that the portfolio of three assets, bubble asset, non-bubble asset and the risk-free asset which will maximize his expected utility of end-of-period wealth. On the other hand, noise traders maximize their random utility[3] of the discrete choice, that is, holding bubble stocks or holding the risk-free asset. We assume that a noise trader's

[3] The qualitative choice models based on maximization of the agent's random utility has been developed by McFadden (1974) .

decision-making is influenced by (i) the decisions of the other noise-traders, and (ii) his expected return momentum on the bubble asset which is defined as the exponential moving average of the price changes. That is, the noise-traders adapt the positive feed-back strategy (momentum strategy) on the bubble asset. To model the interaction among noise-traders, the theory of discrete choice with social interactions proposed by Brock and Durlauf (1999, 2001) is applied. We demonstrate that as the interaction among noise traders (the extent that each noise-trader is influenced by the decisions of other noise-traders,) is strengthened, noise traders begin to *follow the herd*. The noise-traders' herd behavior gives cause to a bubble, and their momentum trading prolongs bubble. In the second half of bubble, rising stock price slows down as the noise-trader's behavior approaches to a stationary state, so that the price momentum begins to decrease. The model demonstrates that decreasing the price momentum lead necessarily to market crash.

The paper proceeds as follows. The model is described in Section 2. In Section 3, we give a theoretical explanation on a mechanism of bubble and crash. We give concluding remarks in Section 5.

2. Model

Consider a market on which two risky assets and a risk-free asset are traded. We divided the risky assets are into two classes: the bubble asset labeled 1 and the non-bubble asset labeled 2, which are portfolios of assets in corresponding sectors. We also divide into two groups of investors with different decision making. The first group of investors is a group of rational investors who maximize their expected utility of wealth in the next period. The second group of investors is the group of the noise-traders who maximize the random utility of the binary choice: holding the bubble asset or the risk-free asset.

2.1 Rational investors

Let us consider the behavior of rational-investors. We assume that there is a number M of rational investors. The object of the rational investors is to maximize the expected utility $EU(W_{t+1})$ of their wealth W_{t+1} in the next period, t+1 by selecting a portfolio mix

of the two types of the risky assets, labeled 1 and 2, and the risk-free asset labeled *f*. We assume that rational investor's preferences are characterized by the constant-absolute risk aversion (CARA) utility with the coefficient of risk aversion, γ. The rational investors are assumed to be identical. We consider the behavior of the representative-rational investor hereafter. The maximization problem which the rational investors solve is equivalent to the mean-variance model. That is, in his choice among all the possible portfolios, the rational investor is satisfied to be guided by its expected yields $E(W_t)$ and its variance $V(W_t)$.

$$\begin{aligned} &\underset{x_{1t},x_{2t},x_{ft}}{Max}\ EU(W_{t+1}) = Max\left\{E(W_t) - \frac{\gamma}{2}V(W_t)\right\} \\ &s.t.\quad p_{1t}(x_{1t} - x_{1t-1}) + p_{2t}(x_{2t} - x_{2t-1}) + q(x_{ft} - x_{ft-1}) = 0 \end{aligned} \tag{1}$$

A rational investor's wealth is written as $W_t = (p_{1t}x_{1t} + p_{2t}x_{2t} + qx_{ft})$ where x_{jt} denotes the demand for the asset *j* in the period *t*, and p_{jt} denotes the unit price of the asset *j* during the period *t*, and q denotes the unit price of a risk-free asset. We assume without loss of generality that q is constant for all *t*. The expected value of the wealth $E(W_{t+1})$, and the variance of the wealth and unity is defined as

$$E(W_{t+1}) = x_{1t}E(p_{1t+1}) + x_{2t}E(p_{2t+1}) + qx_{ft},$$
$$V(W_{t+1}) = \sigma_1^2(x_{1t})^2 + \rho^{12}x_{1t}x_{2t} + \rho^{21}x_{2t}x_{1t} + \sigma_2^2(x_{2t})^2$$

where $E(p_{jt+1})$ is the expected value of p_{jt+1} $(j = 1,2)$, σ_j^2 the variance of p_{jt+1}, and ρ^{ij} the covariance of p_{it+1} and p_{jt+1}.

The corresponding first-order conditions are:

$$\begin{aligned} \frac{\partial EU}{\partial x_{1t}} &= E(p_{1t+1}) - \frac{\gamma}{2}[2\sigma_1^2 x_{1t} + \rho^{12}x_{2t} + \rho^{21}x_{2t}] + \mu p_{1t} \\ &= E(p_{1t+1}) - \gamma[\sigma_1^2 x_{1t} + \rho^{12}x_{2t}] + \mu p_{1t} = 0 \\ \frac{\partial EU}{\partial x_{2t}} &= E(p_{2t+1}) - \gamma[\rho^{21}x_{1t} + \sigma_2^2 x_{2t}] + \mu p_{2t} = 0 \\ \frac{\partial EU}{\partial x_{ft}} &= q + \mu q = 0 \end{aligned} \tag{2}$$

$$\frac{\partial EU}{\partial \mu} = p_{1t}(x_{1t} - x_{1t-1}) + p_{2t}(x_{2t} - x_{2t-1}) + q(x_{ft} - x_{ft-1}) = 0$$

where μ denotes the Lagrangian. Demands for the bubble asset and non-bubble asset of the optimal portfolio are:

$$x_{1t} = \frac{1}{|A|\gamma}\{\left(E(p_{1t+1}) - p_{1t}\right)\sigma_2^2 - \left(E(p_{2t+1}) - p_{2t}\right)\rho^{12}\} \tag{3}$$

$$x_{2t} = \frac{1}{|A|\gamma}\{\left(E(p_{2t+1}) - p_{2t}\right)\sigma_1^2 - \left(E(p_{1t+1}) - p_{1t}\right)\rho^{21}\} \tag{4}$$

$$x_{ft} = x_{ft-1} + \frac{p_{1t}}{q}(x_{1t} - x_{1t-1}) + \frac{p_{2t}}{q}(x_{2t} - x_{2t-1}) \tag{5}$$

where $|A| = \begin{vmatrix} \sigma_1^2 & \rho^{12} \\ \rho^{21} & \sigma_2^2 \end{vmatrix}$.

The excess demands for three assets by rational investors are calculated by subtracting x_{jt} from x_{jt+1}

$$\begin{cases} x_{1t} - x_{1t-1} = \dfrac{1}{\gamma|A|}\{\left(E(p_{1t+1}) - E(p_{1t}) - (p_{1t} - p_{1t-1})\right)\sigma_2^2 - \left(E(p_{2t+1}) - E(p_{2t}) - (p_{2t} - p_{2t-1})\right)\rho^{12}\} \\ x_{2t} - x_{2t-1} = \dfrac{1}{\gamma|A|}\{\left(E(p_{2t+1}) - E(p_{2t}) - (p_{2t} - p_{2t-1})\right)\sigma_1^2 - \left(E(p_{1t+1}) - E(p_{1t}) - (p_{1t} - p_{1t-1})\right)\rho^{21}\} \\ x_{ft} - x_{ft-1} = \dfrac{p_{1t}}{q}(x_{1t} - x_{1t-1}) + \dfrac{p_{2t}}{q}(x_{2t} - x_{2t-1}) \end{cases} \tag{6}$$

The aggregated excess demands for the risky assets by rational investors are obtained by multiplying the number M of rational investors:

$$\begin{cases} M\Delta x_{1t} = \dfrac{M}{\gamma|A|}[\sigma_2^2(E(\Delta p_{1t+1}) - \Delta p_{1t}) - \rho^{12}(E(\Delta p_{2t+1}) - \Delta p_{2t})] \\ M\Delta x_{2t} = \dfrac{M}{\gamma|A|}[\sigma_1^2(E(\Delta p_{2t+1}) - \Delta p_{2t}) - \rho^{21}(E(\Delta p_{1t+1}) - \Delta p_{1t})] \end{cases} \tag{7}$$

where $\Delta x_{it} = x_{it} - x_{it-1}$, $\Delta p_{jt+1} = p_{jt+1} - p_{jt}$ and $E(\Delta p_{jt+1}) = E(p_{jt+1}) - E(p_{jt})$

2.2. Noise traders

We consider the problem of a noise trader's choice between two assets, that is, the bubble asset and the risk free asset. We consider that a noise trader's choice is influenced from the other noise-traders' choice, especially his choice has a tendency to be in favor of the majority decision as Keynes's beauty contest metaphor. To formalize the behavior of a noise-trader, we utilize that the theory of discrete choice with social interactions proposed by Brock and Durlauf (1999, 2001). We assume that there is a number N of noise traders who is inexperienced and have not yet directly experienced the consequences of a market downturn. Individual noise-trader is indexed by *i*. Each noise-trader is assumed to choose the bubble asset or the risk-free asset for each period. We assume that the noise trader only focuses attention on an exchange of the bubble asset and the risk free asset. The noise-trader *i*'s choice at time *t* is $s_{i,t}$ with associated support $\{-1,1\}$. That is, If noise-trader *i* chooses holding of the bubble asset, then $s_{i,t}=1$, and if he chooses holding of the risk-free asset, then $s_{i,t}=-1$.

We consider that the individual noise-trader maximizes the random utility a noise trader receives form holding of an asset.

$$\max_{s_{i,t}\in\{-1,+1\}} U(s_{i,t})$$

The noise trader *i* calculates a value, *U* when he selects each of two assets, that is, the bubble stocks (labeled +) and the risk-free asset (labeled－), and chooses an asset with the higher utility[4]. We assume that the noise-trader's random utility function *U* is decomposed into three components[5]

$$U = s_{i,t}(\lambda \bar{s}_{i,t}^{e} + H_{i,t}) + \varepsilon(s_{i,t}) . \qquad (8)$$

[4] The qualitative choice models based on maximization of the agent's random utility function were developed by McFadden (1974).

[5] The utility function (9) is in line with the general model of discrete choice in the presence of social interactions proposed by Brock and Durlauf (1999, 2000). See also Lux (1995) and Kaizoji (2000).

where $\bar{s}_{i,t}^{e} = \sum_{j \neq i} s_{ij,t}^{e} / (N-1)$. $s_{ij,t}^{e}$ denotes the subjective expected value from the perspective of noise-trader *i* of noise-trader *j* choice at time *t*, so that $\bar{s}_{i,t}^{e}$ denotes the mean value of all noise traders' behavior other than noise-trader *i*'s behavior perceived by noise-trader *i*. The parameter λ measures the degree of dependence across noise-traders, the so-called *conformity effect*. Given λ is a positive, an increase in $\bar{s}_{i,t}^{e}$ raises his utility of holding the bubble asset, and lowers his utility of holding the risk-free asset. The second term H_t [6]denotes the expected excess return or *the expected return momentum* of the bubble asset which is anticipated by noise-traders. The return momentum is defined as the exponential moving average of the risk premium of the bubble asset,

$$H_t = \theta \sum_{k=0}^{\infty} (1-\theta)^k (r_{t-k} - r_f) . \tag{9}$$

where The term $r_{1,t} - r_f = \dfrac{p_{1,t} - p_{1,t-1}}{p_{1,t-1}}$ denotes the realized risk premium of the bubble assets. Note that the return of the risk free asset r_f is assumed to be equal to zero. The return momentum H_t of the bubble asset is equivalent to

$$H_t = (1-\theta) H_{t-1} + \theta (r_{1,t-1} - r_f), \quad H_{t=0} = H_0 \tag{10}$$

where $0 < \theta < 1$. It means that the noise-trader *i*'s expectation on the risk premium is *adaptive*. As the return momentum $H_{i,t}$ of the bubble asset is higher, his utility of holding the bubble asset raises, and the utility of holding the risk-free asset is reduced. The equations (10), and (11) mean that the noise-traders adapt momentum strategies which are a strategy that buys assets with high capital gains and sells assets with poor capital gains over the previous periods[7].

Finally, the term $\varepsilon(s_{i,t})$ denotes the noise-trader *i*'s random utility. One possible explanations for the random term is that there are unobserved characteristics of the individual noise-trader on trading strategies and unobserved attributes of the assets

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[6] $H_{i,t}$ corresponds to the private utility in the words of Brock and Durlauf (2001).

[7] The fact that momentum strategies yield significant profits have been well investigated. Jegadeesh and Titman (1993) examines a variety of momentum strategies and documents that momentum strategies earn profits.

perceived by individual noise-trader independently. The random term may let noise-trader *i* to make different choice with that of the noise-traders who has the same the two deterministic utility terms as him. The term $\varepsilon(s_i)$ is assumed to be independently and identically distributed across noise-traders with the *Gumbell distribution*[8]

$$\Pr[\varepsilon(s_i) \le \varepsilon] = \exp[-\exp[-\varepsilon]]. \tag{11}$$

The probability that a noise trader chooses the bubble asset and the risk free asset is given respectively as :

$$\begin{cases} P^+ = \Pr[U^+ > U^-] = \Pr[\bar{U}^+ - \bar{U}^- \ge \varepsilon(-1) - \varepsilon(+1)] \\ P^- = \Pr[U^- > U^+] = \Pr[\bar{U}^- - \bar{U}^+ \ge \varepsilon(+1) - \varepsilon(-1)] \end{cases} \tag{12}$$

where U^+ denotes the utility U when the noise-trader selects the bubble stocks ($s_{i,t} = +1$) and U^- the utility when the noise-trader selects the risk-free asset ($s_{i,t} = -1$), and,

$$\begin{cases} \bar{U}_t^+ = \lambda \bar{s}_{i,t}^e + H_t \\ \bar{U}_t^- = -(\lambda \bar{s}_{i,t}^e + H_t) \end{cases} \tag{13}$$

and $P_+ + P_- = 1$.

The probability that a utility-maximizing noise trader will selects an asset, is expressed as[9]:

$$\begin{cases} P_t^+ = \dfrac{\exp[\bar{U}_t^+]}{\exp[\bar{U}_t^+] + \exp[\bar{U}_t^-]} \\ P_t^- = \dfrac{\exp[\bar{U}_t^-]}{\exp[\bar{U}_t^+] + \exp[\bar{U}_t^-]} \end{cases} \tag{14}$$

The effects of λ and H_t on the probabilities can be described as follows:

i) As the positive parameter λ pulls up, the noise-traders' conformity effect is enforced.

ii) As the variable H_t increases (decreases), the probability that a noise trader holds the bubble asset rises (falls), and the probability of holding the risk-free asset falls (rises), and vice versa for H_t .

[8] For simplicity of analysis, we set thethe scale factor for a Gumbel probability distribution of the random utility error to unity and reduce the distribution to simplest form (11).

[9] For mathematical derivations of (13), see, for example, Macfadden (1974) and Ben-Akiva, and Lerman (1985).

2.2.1. The self-consistent equilibrium

The expectation of noise-trader *i*'s choice, conditional on his belief concerning the behavior of all noise-traders other than him, can be written as

$$\begin{aligned} E(s_{i,t}) &= +1\cdot\frac{\exp[\bar{U}_t^+]}{\exp\left[\bar{U}_t^+\right]+\exp\left[\bar{U}_t^-\right]} - 1\cdot\frac{\exp[\bar{U}_t^-]}{\exp\left[\bar{U}_t^+\right]+\exp\left[\bar{U}_t^-\right]} \\ &= \tanh\left[H_t + \sum\nolimits_{j\neq i} s_{ij,t}^e / (N-1)\right] \end{aligned} \tag{15}$$

Following Brock and Durlauf (1999, 2001), we assume rational expectations, that is,

$$s_{ij,t}^e = E(s_{j,t}) \tag{16}$$

for all *i* and *j*. Then, one can prove that there exists a self-consistent equilibrium s^* such that

$$s^* = \tanh(H_t + \lambda s^*) \equiv F(s^*). \tag{17}$$

At the self-consistent equilibrium, it is held that $E(s_{i,t}) = E(s_{j,t}) \quad \forall i, j$. That is, at a self-consistent equilibrium, the common noise-trader's expected value equals the expected value of the average choice for the other noise-traders. The self-consistent equilibrium can be interpreted as a solution of Keynes's beauty-contest metaphor (Keynes (1936)). An implication of his metaphor is that an understanding of financial markets requires an understanding not just of market participants' beliefs, but also an understanding of market participants' beliefs about other market participants' beliefs.

2.2.3. Properties of the self-consistent equilibria

We summarize the properties of the self-consistent equiribria as solutions to equation (17). The expected average choice of the noise-traders depends on λ and H_t. For simplicity of analysis, let us assume that the expected return momentum H_t is a parameter H. The properties of the self-consistent equilibria with respect to λ and H as follows:

i) The case of $0 < \lambda < 1$ and arbitrary H:

There is only one possible self-consistent equilibrium s^{**}. For $H = 0$ and $0 < \lambda < 1$, the only one possible solution is zero. In the case which the noise-traders' conformity effect is weak, herding among noise-traders dose not function. In Figure 1 the graphical solution to (17) is plotted for $\lambda < 1$ and the different values of H. An increase (decrease) in H shifts the curve which draw the transcendental equation (17) upward (downward). Therefore, the self-consistent equilibrium moves from the origin to point s_A^{**} (from the origin to s_B^{**}.

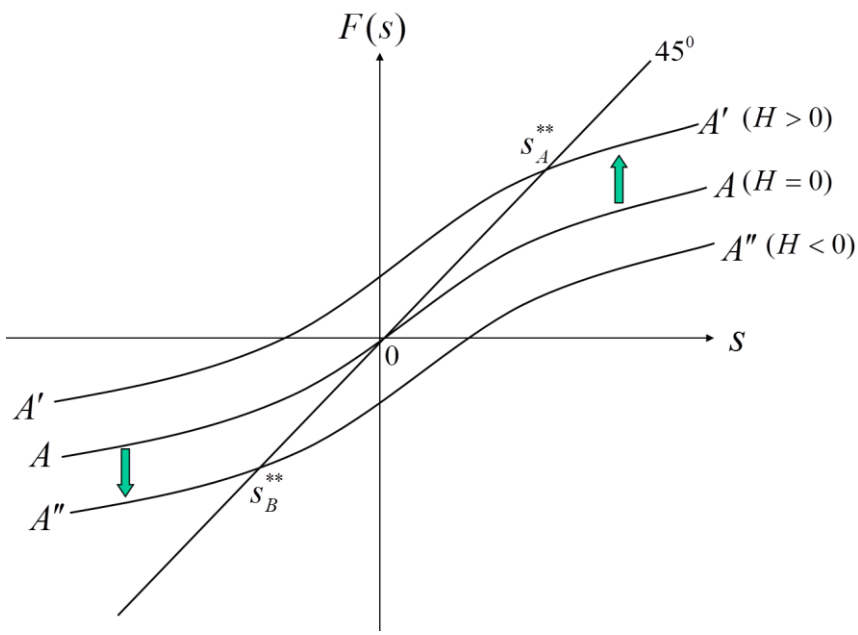

Figure 1: The solutions of the equation (17) for $\lambda < 1$ and the three values of H. There exists a unique self-consistent equilibrium. The straight line is 45 degree line.

ii) The case of $\lambda > 1$ and $|H| < \bar{H}$:

$\bar{H}$ is determined by the equation $\cosh^2[\bar{H} - \sqrt{\lambda(\lambda - 1)}] = \lambda$. Under the conditions, there are three self-consistent equilibria $s^* < s^{**} < s^{***}$. The equilibrium s^* and s^{***} are called the bear-market equilibrium and the bull-market equilibrium respectively. As the parameter λ increases and exceeds

unity, the equilibrium s^{**} is unstable, and appears the bear-market equilibrium $s^{*}(<0)$ and bull-market equilibrium $s^{***}(>0)$ anew. This bifurcation is called as the second-order phase transition that we consider as the origin of a bubble. In Figure 2 the graphical solution to (17) is plotted for $\lambda > 1$ and $H = 0$.

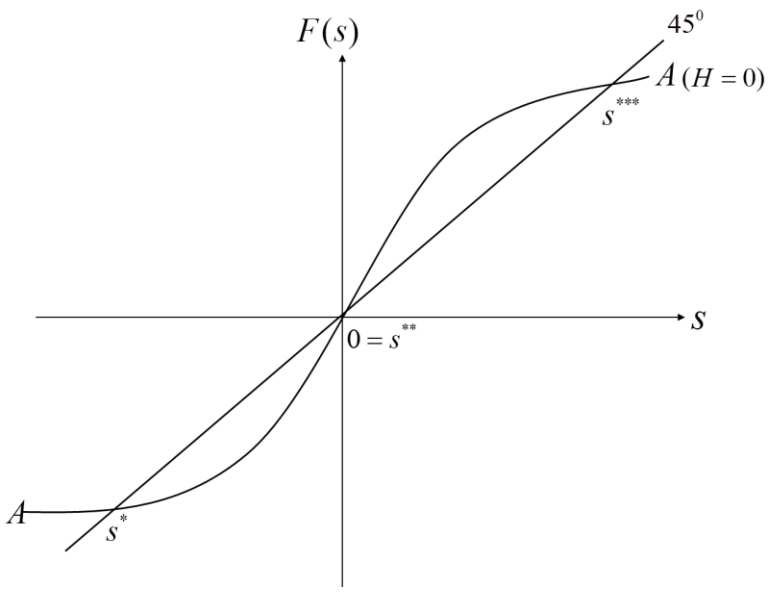

Figure 2: The solutions of the equation (17) for $\lambda > 1$ and $H = 0$. The straight line is 45 degree line. There are the three self-consistent equilibria, s^{*}, s^{**}, and s^{***}.

iii) The case of $\lambda > 1$ and $|H| = \bar{H}$:

Two of the tree solutions $s^{*} < s^{**} < s^{***}$ coincide at $s_c = \pm\sqrt{(\lambda - 1)/\lambda}$.

Given that β is constant, an increase (a decrease) in the return momentum H causes the curve which indicates the transcendental equation (17) to shift up (down), so that the solutions rise (fall). Figure 3 shows the states that two of the tree solutions coincide.

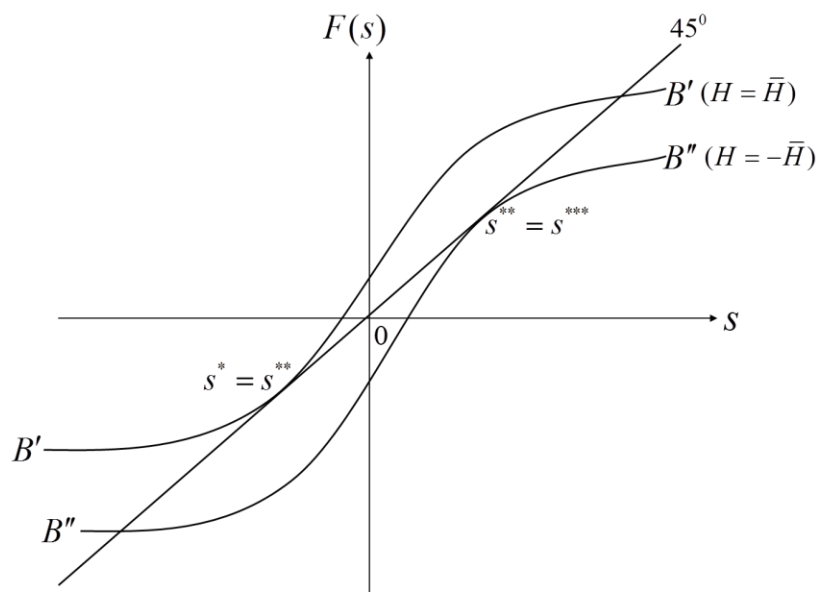

Figure 3: The solutions of the equation (17) for $\lambda > 1$ and $H = \pm\bar{H}$. The straight line is 45 degree line.

iv) There case of $\lambda > 1$ and $|H| > \bar{H}$:

There is one self-consistent equilibrium. When $\lambda > 1$, and H is negative and decreasing continuously, the equilibrium jumps down from s^{***} to s^{*} at the moment that H falls below $-\bar{H}$. Inversely, the self-consistent equilibrium jumps up from s^{*} to s^{***} at the moment that H exceeds $\bar{H}$. This bifurcation is called as the first-order phase transition that is related to market crash in Section 3.

2.2.2. Dynamics

We consider the dynamics of average choice of noise traders at time *t*. We assume that noise-traders' expectation is myopic, that is, $\bar{s}_{i,t}^{e} = \hat{s}_{t-1}, \forall i$ where $s_{t-1} = \sum_{i=1}^{N} \hat{s}_{i,t-1} / N$. We consider the average choice $\hat{s}_t$ is adjusted by the error correction model,

$$\Delta\hat{s}_{t+1} \equiv \hat{s}_{t+1} - \hat{s}_t = \nu[\tanh(\lambda\hat{s}_t + H_t) - \hat{s}_t] \qquad (18)$$

where the right hand side of (17) is the error adjustment term and the parameter λ describes the adjustment speed and is between 0 and 1.

We can rewrite the average choice s_t as $\hat{s}_t = (n_t^+ - n_t^-)/N$ where n_t^+ denotes the number of noise traders who hold the bubble asset at the time *t*, and n_t^- denotes the number of noise traders who hold the risk free asset at time *t*.

Using the difference of s_t from period *t* to period *t+1*, the aggregate excess demand for the bubble asset by all noise traders is defined as

$$Q\left[n_t^+ - n_{t-1}^+\right] = \frac{QN}{2}\left(\hat{s}_t - \hat{s}_{t-1}\right) \tag{19}$$

where the parameter Q denotes the number of shares of the bubble asset which is exchanged in any transaction by a noise trader, and is assumed to be constant. The equation (19) will be utilized when the market prices of the risky assets are calculated under the market clearing conditions in section 2.3.

2.3. Market-clearing prices

The market clearing condition requires that the aggregated excess demand (supply) for each asset by rational investors is equal to the aggregated excess demand (supply) by noise traders from the period *t-1* to the period t. That is, if one noise-trader changes from a holder of the risk-free asset to a holder of the bubble stocks, then the prices are adjusted such that rational investors supply the corresponding number of the bubble stocks. The market clearing conditions are described as

$$\begin{cases} M\Delta x_{1t} + \dfrac{QN}{2}\Delta\hat{s}_t = \dfrac{M}{\gamma|A|}[\sigma_2^2(E(\Delta p_{1t+1}) - \Delta p_{1t}) - \rho^{12}(E(\Delta p_{2t+1}) - \Delta p_{2t})] + \dfrac{QN}{2}\Delta\hat{s}_t = 0 \\ M\Delta x_{2t} = \dfrac{M}{\gamma|A|}[-\rho^{12}(E(\Delta p_{1t+1}) - \Delta p_{1t}) + \sigma_1^2(E(\Delta p_{2t+1}) - \Delta p_{2t})] = 0 \end{cases} \tag{20}$$

Solving the equations (20) with respect to the price changes on the risky assets $\Delta p_{i,t}, (i = 1, 2)$ we can obtain the price changes of the risky assets which satisfy the

market-clearing conditions. In summary, the equilibrium dynamics of stock markets can be described as:

$$\begin{cases} p_{1t} - p_{1t-1} = \gamma \sigma_1^2 \kappa \Delta \hat{s}_t + E(\Delta p_{1t+1}) \\ p_{2t} - p_{2t-1} = \gamma \rho^{12} \kappa \Delta \hat{s}_t + E(\Delta p_{2t+1}) \\ \hat{s}_t - \hat{s}_{t-1} = \nu[\tanh(\beta\lambda \hat{s}_{t-1} + \beta H_t) - \hat{s}_{t-1}] \\ H_t - H_{t-1} = \theta[(r_{1t-1} - r_f) - H_{t-1}] \end{cases} \tag{21}$$

where $\Delta\hat{s}_t = \hat{s}_t - \hat{s}_{t-1}$, and $\kappa = QN/2M$. We assume that the term $E(\Delta p_{jt+1})$, which describes a change of the rational investors' expectations on the *fundamental prices* of the risky assets. The terms are often considered as a random variable. When the noise-trader's conformity effect is weak ($\lambda < 1$), there is a unique equilibrium s^{**}, and the dynamics of $\hat{s}_t$ is globally stable. Under this condition, the dynamics of the risky asset prices are basically driven by the random term $E(\Delta p_{jt+1})$. As Friedman (1953) thought, rational investors counter the deviations of the risky asset prices from fundamentals and so stabilize them. However, when the noise-trader's conformity effect is strong ($\lambda > 1$), the equilibrium s^{**} becomes unstable and the two equilibria, the bull-market equilibrium s^{***} and the bear-market equilibrium s^{*}, are generated anew. As described in detail later on, in this case, the risky asset prices are derived from the fundamental values, and bubbles followed by crashes is caused.

Thereafter, for simplicity of analysis, we consider the terms are constant over time, that is, $E(\Delta p_{jt+1}) = 0, (j = 1, 2)$. Under the assumption, the price changes of the risky assets depend completely on the change $\Delta\hat{s}_t$ of the noise-traders' average choice. Since $\gamma \sigma_1^2 \kappa > 0$, the price change $\Delta p_{1,t} = p_{1,t} - p_{1,t-1}$ of the bubble asset raises (declines) proportionally with respect to $\Delta\hat{s}_t$, and for $\rho^{12} > 0$ (for $\rho^{12} < 0$), the price change Δp_{2t} of the non-bubble asset is positively (negatively) correlated with $\Delta\hat{s}_t$. If one noise-trader changes from a holder of the risk-free asset to a holder of the bubble asset, then the price p_{1t} of the bubble asset rises by $\gamma \sigma_1^2 \kappa > 0$, and the price p_{2t} of the non-bubble asset rises (falls) by $\gamma \rho^{12} \kappa$ for $\rho^{12} > 0$ ($\rho^{12} < 0$).

3. Bubbles and crashes

3.1. How does a bubble come about?

As discussed in subsection 2.3., when the parameter λ exceeds unity, the unique solution $s^{**}(=0)$ is unstable, and appears newly two equilibria, the bear-market equilibrium $s^{*}(<0)$ and bull-market equilibrium $s^{***}(>0)$ under $H_t = 0$. This bifurcation is caused by strengthening of noise-traders' conformity effect. Let us consider the motion of the prices of the bubble asset. Starting from a positive value of $\hat{s}_0$ at the initial time near the unstable equilibrium $s^{**}(=0)$, $\hat{s}_t$ raises toward the bull market equilibrium $s^{***}(>0)$, and it increases the noise-traders' excess demand $(QN\Delta\hat{s}_t / 2)$ for the bubble asset. Run-up in the bubble-asset price p_{1t} is due to an increase in the noise-traders' excess demand $(QN\Delta\hat{s}_t / 2)$ for the bubble asset, and then it increases the expected return momentum H_t. An increase in H_t shifts the curve of hyperbolic tangent (17) upward. The bull-market equilibrium moves to point A to point B in Figure 4. The aggregate demand for the bubble asset by noise-traders is increased further by an increase in the level of $\hat{s}_t$. This inflationary spiral gives cause to the asset bubble. Since for $\lambda > 1$, the bull-market equilibrium $s^{***}(>0)$ moves upward as the momentum H_t rises, the price dynamics (21) is unstable. For $H_t > \bar{H}$, the bear-market equilibrium $s^{*}(<0)$ disappears, and the bull-market equilibrium s^{***} is unique. (See point C in Figure 4.) The bubble persists as long as the imbalance of buyers and sellers over the noise traders is expanding, and the average choice level $\hat{s}_t$ of noise-traders is approaching to the bull-market equilibrium s^{***}.

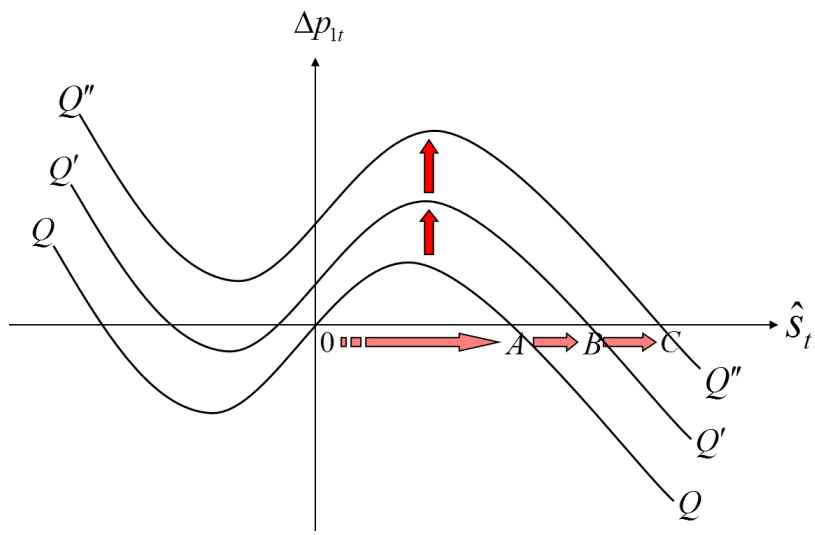

Figure 4: A mechanism of bubble of the risky asset price from the asset price dynamics of (21) for $\lambda > 1$ and $E(\Delta p_{jt+1}) = 0$.

3.2. An explanation of equity premium puzzle

Many recent academic articles have argued that after the increases in stock prices over the last decade, the expected equity premium is low and perhaps negative (Lee, Myers and Swaminathan (1999); Fama and French (2000); Shiller (2000)). Our model gives a persuasive explanation on the *equity premium puzzle*. The expected risk premium is described as

$$E[r_{1t+1}] - r_f = \left(\frac{E(p_{1t+1}) - p_{1t}}{p_{1t}} \right), \text{ and } E[r_{2t+1}] - r_f = \left(\frac{E(p_{2t}) - p_{2t}}{p_{2t}} \right). \tag{22}$$

where $E[r_{jt+1}]$ denotes the expected return of the risky asset *j*, and r_f denotes the return of the risk-free asset that is assumed to be equal to zero.

The price p_{1t} of the bubble asset is over-evaluated against the fundamentals, that is, a negative expected equity premium $\left(E(p_{1t+1}) - p_{1t}\right) / p_{1t}$, in the period of the inflationary spiral. Thus, rational investors sell the bubble asset, and in the opposite direction, more noise traders buy the bubble asset due to rises in their utility of holding the bubble asset. In other words, noise-traders try to surf bravely a bubble whereas rational investors try to avoid cautiously a bubble. Noise traders get a capital gain from

the price run-up of the bubble asset in the period of bubbles. Rational investors suffer capital loss because the actual price change Δp_{1t} of the bubble asset is contrary to the rational investors' expectation in the period of bubbles. In brief, the actual high excessive return on the bubble asset is earned as the result of negative expected returns in the period of bubbles.

3.3. Why does a bubble burst?

In the first half of bubbles, the noise-traders' excess demand for the bubble asset is sharply increasing, so that the price of the bubble asset is also sharply increasing, but in the second half of bubbles, as the noise-traders' imbalance $\hat{s}_t$ is necessarily approaching the bull market equilibrium s^{***} because the imbalance has the upper limit of the unity. Then, the noise-traders' excess demand for the bubble asset is approaching gradually zero, and so a rise in the bubble-asset price slows down. This lowers the expected return momentum H_t bit by bit. Decreasing the expected return momentum H_t move the bull-market equilibrium s^{***} downward, so that the noise traders' excess demand for the bubble asset declines, and at the same time, the bubble-asset price decreases. Still more, a decrease in the bubble asset price decreases the expected return momentum H_t again. In this way, the bubble-asset price necessarily begins to decrease as a reaction of the bubble. This deflationary spiral continues until the expected return momentum H_t declines by the critical value, $-\bar{H}$ at which the bull-market equilibrium s^{***} disappears. In the next instant when H_t falls below $-\bar{H}$, the bear-market equilibrium s^{*} become a unique equilibrium and a market crash can be suddenly caused. The market crash in our model is considered as the first-order phase transition which is a kind of transformation of a thermodynamic system from one phase to another. In our model, the noise-traders' panic selling of the bubble asset is caused by that the utility of the noise trader's selling the bubble asset is progressively higher than that of the noise trader's buying the bubble asset. (See Figure 5.) After a crash, the rational investors buy the bubble asset, which they sell and/or go

short in the period of bubble, back at a low price at the point the expected equity premium (22) is positive.

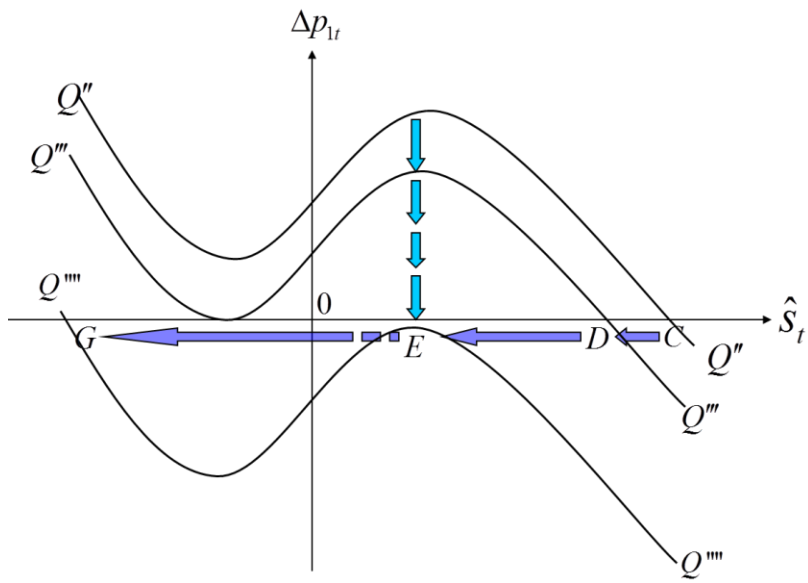

Figure 5: Figure 4: A mechanism of burst of bubble of the risky asset price from the asset price dynamics of (21) for $\lambda > 1$ and $E(\Delta p_{jt+1}) = 0$.

3.4. Contagious effect of bubble

We consider the case that the bubble asset has a weak positive correlation with the non-bubble asset, $\rho^{12} > 0$. As the noise-traders' excess demand for the bubble asset increases, the price of the non-bubble asset p_{2t} increases to some extent reflecting the weak positive-correlation. A crash of the bubble asset also gives rise to the decline in the price of non-bubble asset. The price of non-bubble asset adjusts to reflect the positive correlation, $\rho^{12} > 0$. This example indicates a possibility that a bubble and its subsequent crash is a "contagious" market-wide phenomenon.

3.5. Cycles of bubble and crash

Once a market crash occurs, the market downturn continues until the average choice level $\hat{s}_t$ of noise-traders approaches to the bear-market equilibrium s^*. As $\hat{s}_t$ of noise-traders approaches to the bear-market equilibrium s^*, the noise-traders' excess supply for the bubble asset is approaching gradually zero. Then, a fall in the bubble-asset price slows down. This enhances the value of the expected return momentum H_t. Increasing the expected return momentum H_t moves the bear-market equilibrium s^* upward, so that the noise traders' excess demand for the bubble asset increases, and the bubble-asset price rises. Next, a rise in the bubble asset price increases the expected return momentum H_t. The bubble-asset price necessarily begins to rise as a reaction of the bubble, again. When the expected return momentum H_t rises by the critical value, $\bar{H}$, the bear-market equilibrium s^{***} disappears. When H_t exceeds $\bar{H}$, the bull-market equilibrium s^* become a unique equilibrium, and the risky asset prices enter a bubble phase. In this way, the above process of the bubble and the burst phase of the bubble are repeated. Figure 6 shows the cycle on bubble and crash. The dynamic process which is described above is summarized in the following proposition.

Proposition 1:

The dynamics of the risky asset prices (21) is globally unstable given that the parameter λ, which describes the noise-traders' conformity effect, is greater than unity, the dynamics of the risky asset prices are globally unstable, and the bubble and crash cycles of risky assets, which are described above, are continually repeated.

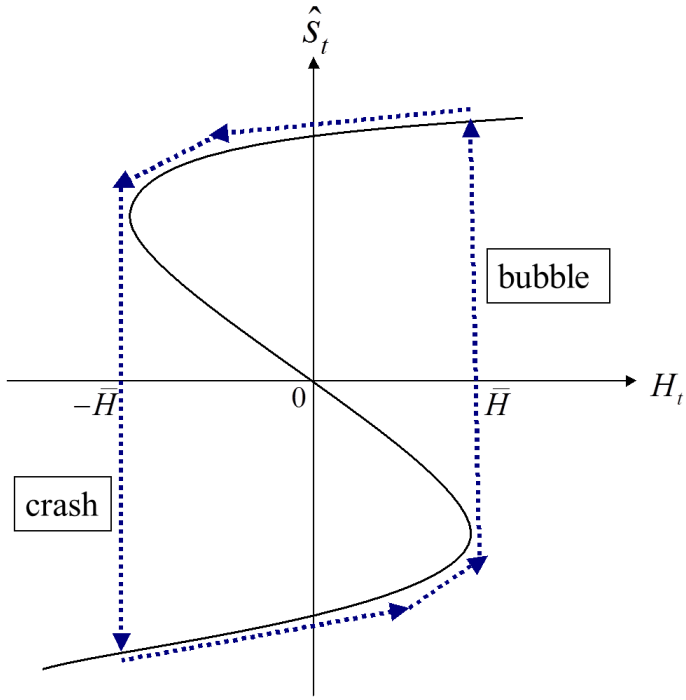

Figure 6: A cycle of bubble and crash appears from the asset price dynamics of (21) for $\lambda > 1$ and $E(\Delta p_{jt+1}) = 0$.

5. Concluding Remarks

This paper provides a new model that gives one potential theoretical explanation for asset bubble followed by crash. We consider the two groups of investors, which have the different sort of rationality regarding their decision making respectively. One is a group of rational investors who employ the Sharpe-Lintnter CAPM which their demands for shares depend on their assessment of fundamental value, and maximize their expected utility of wealth. Another is a group of noise-traders whose demand for bubble asset depend on their expectations on the average value of other noise-traders' investment. The noise trader's behavior is modeled in a framework of the self-consistent equilibrium which can be considered as a modeling of Keynse's beauty contest metaphor. We elucidate a mechanism that (i) noise-traders' herd behavior gives cause to a bubble, and (ii) their momentum trading prolongs bubble, (iii) a bubble is necessarily ended up with a crash, and (iv) the cycles of bubble and crash are repeated.

6. Acknowledgement

The author deeply grateful to Professor Didier Sornette (ETH Zurich) and Professor Alexander Saichev (ETH Zurich) for their valuable comments and discussion on this paper. This research was supported in part by Grant-in Aid for Scientific Research (C) 20510142 from JSPS and the Research Grant from Nomura Foundation, and Momura Foundation.

7. References:

[1] Abreu, D., and M. K. Brunnermeier, (2003), Bubbles and Crashes, Econometrica 71, 173-204.

[2] Allen, Franklin, Stephen Morris, and Hyun Song Shin (2006), Beauty Contests and Iterated Expectations in Asset Markets, Review of Financial Studies 19 no.3, 719-752.

[3] Angeletos, George-Marios, Guido Lorenzoni, and Alessandro Pavan (2010), Beauty Contests and Irrational Exuberance: a Neoclassical Approach, NBER Working Paper No. 15883.

[4] Battalio, Robert and P. Shultz (2006), Options and the Bubble, Journal of Finance 59-5 2017-2102.

[5] Ben-Akiva, M. and S. R. Lerman, (1985) Discrete Choice Analysis: Theory and Application to Travel Demand, The MIT Press.

[6] Black, F., 1986, Noise, The Journal of Finance, 41 (3), Papers and Proceedings of the Forty-Fourth Annual Meeting of the America Finance Association, New York, New York, December 28-30, 1985. (Jul., 1986), 529-543.

[7] Biais, B., and P. Bossaerts, (1998), Asset Prices and Trading Volume in a Beauty Contest, Review of Economic Studies 65, 307–340.

[8] Brennan, Michael, (2004), How Did It Happen?, Economic Notes 33, 3-22.

[9] Brock, William A., and Durlauf, Steven N. (1999), A formal model of theory choice in science, Economic Theory 4, 113-130 (1999).

[10] Brock, William A., and Durlauf, Steven N. (2001), Discrete choice with social interactions, The Review of Economic Studies, 68, No.2, 235-260.

[11] Brunnermeier, Markus and Stefan Nagel, (2004), Hedge funds and the technology bubble, Journal of Finance, 59 (5), 2013 - 2040.

[12] Chen, Joseph, Harrison Hong, and Jeremy Stein, 2002, Breadth of ownership and stock returns, Journal of Financial Economics 66, 171-205.

[13] DeBondt, W. F.M., and Richard Thaler, (1985), Does the stock market overreact? Journal of Finance 40, 793-808.

[14] De Long, B. J., A. Shleifer; L. H. Summers; R. J. Waldmann, 1990a, Positive Feedback Investment Strategies and Destabilizing Rational Speculation, The Journal of Finance, 45 (2), 379-395.

[15] DeLong, J. B., Andrei Shleifer, Lawrence Summers and Robert Waldmann (1990), Noise trader risk in financial markets, Journal of Political Economy 98, 703-38.

[16] Fama, Eugene and Kenneth French. (2002) “The Equity Premium,” Journal of Finance 57, 2, (2002) 637-659.

[17] Harrison, J.M. and D.M. Kreps, 1978, Speculative Investor Behavior in a Stock Market with Heterogeneous Expectations, The Quarterly Journal of Economics 92(2), 323-336.

[18] Hong, H. and J. Stein, 1999, A Unified Theory of Underreaction, Momentum Trading, and Overreaction in Asset Markets, Differences of opinion, shortsales constraints and market crashes, The Journal of Finance, Vol. 54, No. 6. (Dec., 1999), pp. 2143-2184.

[19] Hong, H. and J. Stein, 2003, Differences of opinion, short-sales constraints and market crashes, Review of Financial Studies 16, 487-525.

[20] Hong, Harrison, Jose Scheinkman, and Wei Xiong, 2006, Asset Float and Speculative Bubbles, Journal of Finance 59 (3), 1073-1117.

[21] Jegadeesh, N., and S. Titman, 1993, "Returns to buying winners and selling losers: Implications for stock market efficiency," Journal of Finance 48, 65-91.

[22] Kaizoji, Taisei, Speculative bubbles and crashes in stock markets: an interacting agent model of speculative activity, Physica A. 287, 3-4 (2000), 493-506.

[23] Keynes, John Maynard (1936), The General Theory of Employment, Interest and Money, Macmillan, London.

[24] Kyle, Albert S., 1985, Continuous Auctions and Insider Trading, Econometrica 53, 1315-1335.

[25] Lintner, John (1969), The Aggregation of Investor's Diverse Judgments and Preferences in Purely Competitive Security Markets, The Journal of Financial and Quantitative Analysis, Vol. 4, No. 4., pp. 347-400.

[26] Lee, Charles M. C., James Myers and Bhaskaran Swaminathan (1999), “What is the Intrinsic Value of the Dow?” Journal of Finance. 54, pp.1693–1741.

[27] Lux, Thomas, Herd behavior, bubbles and crashes, The Economic Journal, Vol. 105, No. 431 pp. 881-896.

[28] McFadden, Daniel, (1974) "Conditional Logit Analysis and Qualitative Choice behavior," in Paul Zarembka (ed.), Frontiers in Econometrics (New York: Academic Press).

[29] Miller, Edward, 1977, Risk, uncertainty and divergence of opinion, Journal of Finance 32, 1151-1168.

[30] Mossin, J. (1966), Equilibrium in a Capital sset Market, Econometrica, Vol. 34, No. 4, pp. 768-783.

[31] Haruvy, Ernan, E., Lahav, Y., and C. Noussair (2007), Traders’ Expectations in Asset Markets: Experimental Evidence, forthcoming in the American Economic Review.

[32] Smith, V. L., Suchanek, G. L., and Williams, A. W., (1988), Bubbles, Crashes and Endogenous Expectations in Experimental Spot Asset Markets, Econometrica, 56(5), pp. 1119-1151.

[33] Ofek, Eli, and Matthew Richardson, 2003, DotCom Mania: The rise and fall of Internet stock prices, Journal of Finance 58, 1113-1137.

[34] Greenwood, Robin, and Stefan Nagel, (2008), Inexperienced Investors and Bubbles, Working Paper 14111, and Journal of Financial Economics (forthcoming).