Optimal dynamic demands in commodity futures markets with a stochastic convenience yield[‡]

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Abstract

The main objective of this paper is to address, in an intertemporal framework, the issue of using storable commodity futures as vehicles for hedging purposes when, in particular, the convenience yield as well as the market prices of risk evolve randomly over time. We derive optimal demands of an unconstrained investor endowed with a CRRA utility function. We suggest various decompositions of this demand allowing an investor to assess the impact of each and every state variable on optimal demands and to specify the role played by each risky asset. In particular, the convenience yield has a strong impact on the speculation and hedging components and its orthogonal risk is hedged by the futures contract. Moreover, optimal demands can be computed in a simple recursive way, which, combined with quasi-analytical solutions, may facilitate the use of our model for practical considerations.

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Keywords: Commodity spot prices; futures prices; convenience yield; stochastic market prices of risk; dynamic optimal demands

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1. Introduction

Futures markets have experienced a dramatic growth, worldwide, of both trading volume and contracts written on a wide range of underlying assets. These features make it easier to use futures contracts as hedging instruments against unfavorable changes in the investment opportunity set, i.e. changes in state variables describing the economic/financial environment. The growing activity of these markets has been accompanied, since the original normal backwardation (Keynes, 1930; Hicks, 1939) and the theory of storage (Kaldor, 1939; Working, 1949; Brennan, 1958), by a substantial body of literature devoted to pricing and hedging with futures contracts. Besides, commodities are considered by fund managers as an alternative asset class to traditional assets such as stocks and bonds. Indeed, commodities are supposed to be good (stocks and bonds) portfolio diversifiers, and efficient hedging instruments against the inflation risk. Although, a significant empirical literature examines these two commodities' properties (see, for example, Erb and Harvey, 2006; Gorton and Rouwenhorst, 2006; Kat and Oomen, 2007 ; Daskalaki and Skiadopoulos, 2011), to the best of our knowledge, there is no paper dealing with the question of how to hedge risks with commodity futures, in particular. The main objective of this paper is to address, in a continuous-time context, the issue of using storable commodity futures, by an unconstrained investor[1], as vehicles for hedging purposes.

An abundant literature has been devoted to pricing commodity futures[2]. The models developed explain the evolution of the futures prices through the random evolution of several relevant state variables. The stochastic processes of these variables are specified exogenously. The convenience yield, in accordance with the theory of storage, turns out to be the crucial variable, which constitutes one of the main differences between spot commodity prices and prices of financial assets. The recent sharp increase and then fall in commodity prices has revived the interest in commodity risk management. Futures contracts are major tools used by investors for hedging in order to mitigate their exposure to changes in commodity prices. Surprisingly, while there are a number of models dealing

[1] The unconstrained investor is allowed to freely trade on the primitive assets, namely the underlying spot asset and, if need be, other risky assets.

[2] See, for instance, Gibson and Schwartz (1990), Schwartz (1997), Hilliard and Reis (1998), Miltersen and Schwartz (1998), Yan (2002), Sorensen (2002), Nielsen and Schwartz (2004), Casassus and Collin-Dufresne (2005) and Chiarella et al. (2009).

with futures hedging, to the best of our knowledge, the specific case of commodity futures contracts with a stochastic convenience yield has not yet been addressed in the relevant literature[3]. However, it is widely recognised that the convenience yield evolves randomly over time resulting in a stochastic investment opportunity set. Thus, in a multi period setting, non-myopic investors have a demand for intertemporal hedging stemming from the convenience yield risk. Moreover, a growing number of empirical studies on commodity return predictability stress the important role of the convenience yield, in particular (see, for instance, Fama and French, 1987, 1988; Besembinder and Chan, 1992; Khan et al., 2007; Hong and Yogo, 2009). In addition, the evidence of predictability is consistent with time-varying risk premiums in commodities. In our environment, to the extent that spot commodity prices, futures prices and inventory decisions are related (see, for example, Brennan, 1958; Litzenberger, Rabinowitz 1995; Routledge et al., 2000), we would expect market prices of risk to be stochastic. This is in line with some papers studying optimal asset allocation with stochastic prices of risk (see, for example, Kim and Omberg, 1996; Lioui and Poncet, 2001; Brennan and Xia, 2002; Wachter, 2002; Detemple et al., 2003; Sangvinatsos and Wachter, 2005; Lioui, 2007; Liu, 2007).

This paper provides a model of optimal demand that could better account for the way both the stochastic convenience yield and stochastic (affine) market prices of risk affect the optimal demand of an unconstrained investor[4]. In order to do so, the economic framework retains the spot commodity price, the instantaneous interest rate and the convenience yield as the relevant imperfectly correlated mean-reverting state variables associated with the dynamics of the futures price[5]. The optimal demand for commodity futures contracts is derived for an investor who maximizes the expected constant

[3] An exception is Hong (2001) whose economic environment and objective differ considerably from ours in that he examined the impact of a stochastic convenience yield on the term structure of open interest, i.e., the total number of contracts outstanding.

[4] Other theoretical models examining dynamic asset allocation with futures contracts (see, among others, Ho, 1984; Stulz, 1984; Adler and Detemple, 1988a, b; Duffie and Jackson, 1990; Briys et al., 1990; Duffie and Richardson, 1991; Lioui et al., 1996) deal with a constraint utility maximizer investor.

[5] As the goal of this paper is not the valuation of futures contracts per se, we follow the reference models in the literature, Schwartz (1997), Hilliard and Reis (1998) and Casassus and Collin-Dufresne (2005), and choose these three commonly used and well identified variables. However, the setting can be extended to a multidimensional state variables space.

relative risk aversion (CRRA) utility function of her (his) lifetime consumption and final wealth by following the no-arbitrage martingale approach (Karatzas, Lehoczky and Shreve, 1987; Cox and Huang, 1989). This framework takes into account the main characteristics of commodity markets and provides explicit solutions up to resolution of ordinary differential equation (ODEs) (see Liu, 2007) for the optimal unconstrained investor's demand, which is classically composed of a speculative part and of a hedging term.

A thorough study of the speculative and of the hedging components allows us to enrich the analysis of optimal demands by going beyond the existing studies by suggesting various decompositions. This is accomplished by introducing into the economic framework two synthetic assets replicating the orthogonal sources of risk of the interest rate and the convenience yield[6]. Usually, in a continuous-time framework, papers obtain general formulae for these two components without deriving specific formulae for each asset. In the case of futures contracts for a constrained investor, Adler and Detemple (1988 a, b) suggested expressions for the futures contract and the spot. We generalize this result along the lines of our framework by deducing the individual speculative and hedge proportions invested in the spot commodity, a discount bond and the futures contract, which may be computed in a useful recursive way underlying the interactions between risky assets demands. The position, short or long, in each asset, may therefore be easily calculated and the economic importance of some commodity markets features may be examined. In particular, simulations reveal that mean-reversion in the state variables and in the prices of risk as well as the correlation between the state variables determine the sign of the speculative and hedging positions. Moreover, the volatility of the convenience yield and the price of risk associated with it appear to play a prominent role in determining these positions.

As a consequence of the calculation of the individual proportions for each asset, our analysis clarifies the role played by the primitive assets and the futures contract when speculating and hedging.

[6] An orthogonal source of risk is related to the construction of correlated Brownian motions. It corresponds to the Brownian motions associated with the short rate and the convenience yield, which are not correlated with the spot commodity price.

Our analysis also calls into question Breeden (1984) result[7] by assigning primitive assets a specific task: hedging the orthogonal risk of the (log) spot commodity and the short rate. Besides, the orthogonal risk associated with the convenience yield is uniquely hedged by the futures contract.

An important question is to know which state variables should be included in the investment opportunity set and how they affect optimal demands. The relevant literature has not explored this question. In our model, the investor is able to assess the influence of the state variables on her (his) optimal demand and can therefore rule on the relevance of the investment opportunity set. Indeed, the impact of the state variables on the optimal hedging proportions is measured through the sensitivity of an investor-specific bond on the state variables. Especially, simulations show that the convenience yield has a strong effect on the speculative proportion and on the hedging proportions as well for the three risky assets, while the effect of the short rate is less pronounced.

The remainder of the paper is organized as follows. In section 2, the economic framework is described and the investor's optimization problem is formulated. Section 3 is devoted to the derivation of the optimal asset allocation for the unconstrained investor. An illustration of the behavior of this demand, via a numerical example, is given in section 4. Section 5 offers some concluding remarks and suggests some potential future extensions. All the proofs have been gathered in the Appendix.

2. The general economic framework

Consider a continuous-time frictionless economy. The uncertainty in the economy is represented by a complete probability space (Ω, F, P) with a standard filtration $F = \{F_t : t \in [0,T]\}$, a finite time period [0, T] with $T > 0$, the historical probability measure P and a 3-dimensional vector of independent standard Brownian motions, $z(t)' = (z_S(t), z_u(t), z_v(t))$, defined on (Ω, F), where $'$ stands for the transpose.

[7] Only one model (Breeden, 1984) studied optimal hedging in commodity futures markets (the convenience yield is not modelled) in the case of an unconstrained investor when the futures contracts are written on the state variables and have instantaneous maturity. As a consequence, the primitive assets are ineffective in hedging the risk of the state variables.

In this section, following Schwartz (1997), Hilliard and Reis (1998) and Casassus and Collin-Dufresne (2005), three imperfectly correlated factors are assumed to be associated with the dynamics of the futures prices: the logarithm of spot commodity price, *X(t) = Ln (S(t))*, the instantaneous riskless interest rate, *r(t)*, and the instantaneous convenience yield, $\delta(t)$. In the sequel of the paper, $\lambda_i(.)$ and σ_i stand for the market prices of risk related to the state variables and the strictly positive instantaneous volatility of the state variables respectively, while ρ_{ij}, with $i \neq j$, denotes the correlation coefficient for $i = X(t), r(t), \delta(t)$. Σ_{kl}, with $k \neq l$ represents either the covariance between the assets or between the assets and the state variables. $Y(t)' = [X(t) \quad r(t) \quad \delta(t)]'$ is the vector of the state variables that describes the economy.

X(t) satisfies the following stochastic differential equation (SDE hereafter):

$$dX(t) = \left(r(t) - \delta(t) + \sigma_S \lambda_X (X(t), r(t), \delta(t)) - \frac{1}{2}\sigma_S^2 \right) dt + \sigma_S dz_S(t) \tag{1}$$

with initial condition *LnS(0)≡LnS*.

The short rate is governed by the following stochastic process:

$$dr(t) = (\vartheta(t) - \alpha r(t))dt + \sigma_r [\rho_{sr} dz_S(t) + \rho_{ur} dz_u(t)] \tag{2}$$

with initial condition *r(0)≡r*. The drift in the stochastic process of the short rate is a deterministic function, $\vartheta(t)$, such that the model incorporates all the information present in the current term structure (see Hull and White, 1990; Heath et al., 1992). $\vartheta(t) = \alpha f(0,t) + \frac{\partial f(0,t)}{\partial t} + \sigma_r^2 D_{2\alpha}(t) + \rho_{Sr}\sigma_r \lambda_{X0}$, where *f(0,t)* describes the initial forward yield curve and $D_{2\alpha}(t) = (1 - e^{-2\alpha t})/\alpha$. α is the constant speed of mean reversion of the short rate, λ_{X0} is specified below and $\rho_{ur} = \sqrt{1 - \rho_{Sr}^2}$.

The instantaneous convenience yield evolves stochastically over time by following a mean-reverting process:

$$d\delta(t) = k(\bar{\delta} - \delta(t))dt + \sigma_\delta [\rho_{S\delta} dz_S(t) + \rho_{u\delta} dz_u(t) + \rho_{v\delta} dz_v(t)] \tag{3}$$

with initial condition $\delta(0) \equiv \delta$. The convenience yield has a tendency to revert to a constant long-run

convenience yield, $\bar{\delta}$, with a speed of mean-reversion *k*. Empirical studies (see Fama and French 1988, and Brennan 1991) found that the convenience yield should be specified by a mean-reverting process. $\rho_{u\delta} = \dfrac{\rho_{r\delta} - \rho_{Sr}\rho_{S\delta}}{\rho_{ur}}$ and $\rho_{v\delta} = \dfrac{\sqrt{1-\rho_{Sr}^2 - \rho_{S\delta}^2 - \rho_{r\delta}^2 + 2\rho_{Sr}\rho_{S\delta}\rho_{r\delta}}}{\rho_{ur}}$.

The market prices of risk associated with the state variables are not constant but stochastic and depend on the levels of the state variables. To allow for an analytical tractability of our model, we opt for an affine specification of these prices of risk. Assuming random prices of risk differentiates our model from the vast majority of the models exploring commodity futures pricing (a notable exception is Casassus and Collin-Dufresne, 2005). More significantly, with regard to our study, stochastic prices of risk related to each state variable also distinguish our model from those dealing with dynamic asset allocation and hedging which usually consider only one stochastic price of risk (see also Sangvinatsos and Wachter, 2005; Liu, 2007). This will have key implications on the investor's optimal portfolio rules. To characterise the dependence of the spot price on the level of inventories (see, for instance, Brennan, 1958; Dincerler et al. 2005), the price of risk associated with the (log) of the spot price process is an affine function of the level of both the (log) of the spot price and the convenience yield: $\lambda_X\left(X(t),\delta(t)\right) = \lambda_{X0} + \lambda_{XX}X(t) + \lambda_{X\delta}\delta(t)$. The prices of risk related to the interest rate and the convenience yield are also affine functions: $\lambda_r\left(r(t)\right) = \lambda_{r0} + \lambda_{rr}r(t)$ and $\lambda_\delta\left(\delta(t)\right) = \lambda_{\delta 0} + \lambda_{\delta\delta}\delta(t)$. $\lambda_{X0}, \lambda_{XX}, \lambda_{X\delta}, \lambda_{r0}, \lambda_{rr}, \lambda_{\delta 0}$ and $\lambda_{\delta\delta}$ are constants. *λ(t)* is a stochastic vector of the market prices of risk:

$$\lambda(t) = \begin{pmatrix} \lambda_X(t) \\ -\dfrac{\rho_{Sr}}{\rho_{ur}}\lambda_X(t) + \dfrac{1}{\rho_{ur}}\lambda_r(t) \\ \dfrac{\rho_{Sr}\rho_{u\delta} - \rho_{S\delta}\rho_{ur}}{\rho_{ur}\rho_{v\delta}}\lambda_X(t) - \dfrac{\rho_{u\delta}}{\rho_{ur}\rho_{v\delta}}\lambda_r(t) + \dfrac{1}{\rho_{v\delta}}\lambda_\delta(t) \end{pmatrix} = \lambda_0 + \lambda_Y Y(t) \tag{4}$$

where λ_0 and λ_Y are given in Appendix A.

In addition to the spot commodity, there are in the economy a locally riskless asset, the savings account, such that: $\beta(t) = \exp\left\{\int_0^t r(s)ds\right\}$, with initial condition $\beta(0) = 1$, and two risky traded assets. The first risky security is a discount bond with maturity T_B, whose price, at time *t*, $0 \le t \le T_B$, is

$B(r,t,T_B) \equiv B(t,T_B) = \exp\{-r(t)D_\alpha(t,T_B) + C(t,T_B)\}$. The second additional risky asset is a futures contract written on a commodity with maturity T_H, whose price, at date *t*, $0 \le t \le T_H \le T_B$, $H(Y(t),t,T_H) \equiv H(t,T_H) = \exp\{X(t) - \delta(t)D_\kappa(t,T_H) + r(t)D_\alpha(t,T_H) + K(t,T_H)\}$. $D_x(t,y) = (1-e^{-x(y-t)})/x$ and $C(T_B,T_B)$ and $K(T_H,T_H)$ are deterministic functions[8]. Assuming that the risky securities price functions are twice continuously differentiable in the state variables, their price dynamics can be written as follows:

$$\begin{pmatrix} \dfrac{dS(t)}{S(t)} \\ \dfrac{dB(t,T_B)}{B(t,T_B)} \\ \dfrac{dH(t,T_H)}{H(t,T_H)} \end{pmatrix} = \begin{pmatrix} \mu_S(t) \\ \mu_B(t,T_B) \\ \mu_H(t,T_H) \end{pmatrix} dt + \begin{pmatrix} \sigma_S & 0 & 0 \\ \rho_{Sr}\sigma(t,T_B) & \rho_{ur}\sigma(t,T_B) & 0 \\ \sigma_{HS}(t,T_H) & \sigma_{Hu}(t,T_H) & -\sigma_{Hv}(t,T_H) \end{pmatrix} \begin{pmatrix} dz_S(t) \\ dz_u(t) \\ dz_v(t) \end{pmatrix} \tag{4}$$

$$dV(t) = I_V(t)[\mu(t)dt + \sigma dz(t)]$$

with the initial condition *V(0)≡V*. $V(t) = [S(t) \quad B(t,T_B) \quad H(t,T_H)]'$, $I_V(t)$ is a diagonal matrix, $\mu(t), \mu_S(t), \mu_B(t,T_B)$ and $\mu_H(t,T_H)$ represent the instantaneous expected rate of returns of the vector *V(t)*, the spot price, the discount bond and the futures price respectively. σ is the 3-dimensional volatility matrix, which is of full rank, hence the market is dynamically complete. $\sigma_{HS}(t,T_H) = \sigma_S + \rho_{Sr}\sigma(t,T_H) - \rho_{S\delta}\sigma_\delta D_k(t,T_H)$, $\sigma_{Hu}(t,T_H) = \rho_{ur}\sigma(t,T_H) - \rho_{u\delta}\sigma_\delta D_k(t,T_H)$ and $\sigma_{Hv}(t,T_H) = \sigma_\delta \rho_{v\delta} D_k(t,T_H)$. The volatility of the discount bond, $\sigma(t,T_B) = \sigma_r D_\alpha(t,T_B)$. $\sigma_B(t,T_B)' = [\rho_{sr}\,\sigma(t,T_B) \quad \rho_{ur}\sigma(t,T_B) \quad 0]$ and $\sigma_H(t,T_H)' = [\sigma_{HS}(t,T_H) \quad \sigma_{Hu}(t,T_H) \quad -\sigma_{Hv}(t,T_H)]$ are the diffusion vectors of the discount bond and the futures price respectively.

Since we are interested in futures contracts, the futures price changes are credited to or debited from a margin account with interest at the continuously compounded interest rate *r(t)*. The futures contract is indeed assumed to be marked to market continuously rather than on a daily basis, and then to have always a zero current value. The current value of the margin account, *M(t)*, is then equal to:

$$M(t) = \int_0^t \exp\left\{\int_u^t r(v)dv\right\} \theta_H(u,T_H)dH(u,T_H) \tag{5}$$

[8] There is no need to specify the expression of $C(t,T_B)$ and $K(t,T_H)$, since it will not be used in the rest of the paper.

Applying Itô's lemma to the above equation yields:

$$dM(t) = r(t)M(t)dt + \theta_H(t,T_H)dH(t,T_H) \tag{6}$$

where $\theta_H(t,T_H)$ represents the number of the futures contracts held at time *t*.

The unconstrained investor, endowed with an initial wealth *W(0)*, has an investment horizon T_I, $0 \le t \le T_I \le T_H \le T_B$, and (s)he is endowed with CRRA preferences over consumption and terminal wealth:

$$U(c,W(T_I)) = a\int_t^{T_I} e^{-\eta(s-t)} \frac{c(s)^{1-\gamma}}{1-\gamma} ds + (1-a)e^{-\eta(T_I-t)} \frac{W(T_I)^{1-\gamma}}{1-\gamma} \tag{7}$$

where *U(.)* is a Von Neumann-Morgenstern utility function satisfying the usual Inada conditions, $c(t) \ge 0$ and $W(T_I)$ represent consumption at time *t* and the agent's terminal wealth respectively. η is a subjective discount rate and *a* is the relative weight of the intermediate consumption and the terminal wealth. When $\gamma = 1$, the "reference" utility in the finance literature is obtained, that is, the logarithmic utility function characterizing a Bernoulli investor: $U(c,W(T_I)) = a\int_t^{T_I} e^{-\eta(s-t)} Ln(c(s))ds + (1-a)e^{-\eta(T_I-t)} Ln(W(T_I))$. In this case, the investor behaves myopically in such a way that his (her) optimal demand will not include any component associated with a stochastic opportunity set.

To determine the optimal consumption and asset allocation, each investor maximizes the expected utility function of her (his) lifetime consumption and terminal wealth. The market described above is dynamically complete for $T_I \le \min(T_B, T_H)$, since the number of sources of risk (Brownian motions) is equal to that of the traded risky securities. Karatzas et al. (1987) and Cox and Huang (1989; 1991) used the martingale approach to study the consumption-portfolio problem in a continuous-time setting. Their main idea is to transform this dynamic problem into the following static one:

$$\max_{\{c,W(T_I)\}} \mathrm{E}\left[a\int_t^{T_I} e^{-\eta(s-t)} \frac{c(s)^{1-\gamma}}{1-\gamma} ds + (1-a)e^{-\eta(T_I-t)} \frac{W(T_I)^{1-\gamma}}{1-\gamma} \Big| F_t \right] \tag{8}$$

$$\text{s.t. } \frac{W(t)}{G(t)} = \mathrm{E}\left[\int_t^T \frac{c(s)}{G(s)} ds + \frac{W(T_I)}{G(T_I)} \Big| F_t \right]$$

where $\mathrm{E}[\cdot \mid F_t] \equiv E_t[.]$ denotes the expectation, under *P*, conditional on the information, F_t, available at time *t* and $G(t) = \frac{\beta(t)}{\xi(t)} = \exp\left\{ \int_0^t r(u)du + \frac{1}{2}\int_0^t \|\lambda(u)\|^2 du + \int_0^t \lambda(u)' dz(u) \right\}$, with *G(0) = 1*, represents the numéraire or optimal growth portfolio such that the value of any admissible portfolio relative to this numéraire is a martingale under *P* (see Long, 1990; Merton, 1990; Bajeux-Besnainou and Portait, 1997). $\| \ \|$ stands for the norm in R^3 and $\xi(t)$ is the Radon-Nikodym derivative of the so-called, unique, risk-neutral probability measure *Q* equivalent to the historical probability *P*, such that the relative price (with respect to the savings account chosen as numéraire), of any risky security is a *Q*-martingale (see Harrison and Pliska, 1981).

3. Optimal dynamic strategies

Having described the economic framework, we shall examine the optimal consumption and portfolio strategy problem for our unconstrained investor when the financial market is dynamically complete.

Given the CRRA utility function and the numéraire portfolio $G(t)$, the solution to the static problem (8), which is a standard Lagrangian optimization problem, determines the investor's optimal consumption and wealth at time *t*:

$$c(t)^* = \frac{W(t)^*}{\Phi(\gamma, t, T_I)} \tag{9}$$

$$W(t)^* = \zeta^{-\frac{1}{\gamma}} G(t)^{\frac{1}{\gamma}} \Phi(\gamma, t, T_I) \tag{10}$$

where ζ is the Lagrangian associated with the static program.

$\Phi(\gamma, t, T_I) = a^{\frac{1}{\gamma}} \int_t^{T_I} e^{-\frac{\eta}{\gamma}(s-t)} E_t\left[\left(\frac{G(t)}{G(s)}\right)^{1-\frac{1}{\gamma}}\right] ds + (1-a)^{\frac{1}{\gamma}} e^{-\frac{\eta}{\gamma}(T_I - t)} E_t\left[\left(\frac{G(t)}{G(T_I)}\right)^{1-\frac{1}{\gamma}}\right]$ is the wealth-to-consumption ratio.

The resolution of the expectation in $\Phi(\gamma, t, T_I)$ may be simplified by making an appropriate change of probability measure. This change holds for any diffusion process and for any price of risk. We suggest the measure $P^{(\gamma, T_I)}$, called the CRRA T_I-forward probability measure or the CRRA-forward

measure (see Appendix A). In a recent article, Detemple and Rindisbacher (2009) have substantially generalized the work of Lioui and Poncet (2001) and Munk and Sorensen (2004) by using the forward probability measure (Jamshidian, 1989; Geman et al., 1995) and a zero-coupon bond, $B(t,T_I)$, as numéraire, to derive optimal portfolio rules in a very general setting for non-Markovian processes and for concave non-specified utility functions. To obtain optimal demands, Rodriguez (2002), Stoikov and Zariphopoulou (2005), Björk et al. (2008) and Buraschi et al. (2010) used a change of probability measure related to a CRRA utility function, but in this paper, we attempt to clarify this change of measure. Although, Detemple and Rindisbacher's change of probability measure is different than the CRRA one, our intermediate results are similar to those of these authors and can be considered as a special case of their very general results.

Under $P^{(\gamma,T_I)}$, $\Phi(\gamma,t,T_I)$ can be rewritten in the following way

$$\Phi(\gamma,t,T_I) = a^{\frac{1}{\gamma}} \int_t^{T_I} e^{-\frac{\eta}{\gamma}(s-t)} \varphi(\gamma,t,s)ds + (1-a)^{\frac{1}{\gamma}} e^{-\frac{\eta}{\gamma}(T_I-t)} \varphi(\gamma,t,T_I) \tag{11}$$

where $\varphi(\gamma,t,T_I) = B(t,T_I)^{1-\frac{1}{\gamma}} B_\gamma(t,T_I)$ and $B_\gamma(t,T_I)$ is given by:

$$B_\gamma(Y(t),t,T_I) \equiv B_\gamma(t,T_I) = E_t^{(P^{\gamma,T_I})}\left[\exp\left\{-\int_t^{T_I} \frac{\gamma-1}{2\gamma^2}\left\|\lambda(u)-\sigma_B(u,T_I)\right\|^2 du\right\}\right] \equiv E_t^{(P^{\gamma,T_I})}\left[\exp\left\{-\int_t^{T_I} y_\gamma(u)du\right\}\right] \tag{12}$$

Inspired by the relevant literature of the term structure of interest rates (see Duffie and Kan, 1996; Dai and Singleton, 2000; Ahn et al., 2002), $y_\gamma(t,T_I)$ is a quadratic function and $B_\gamma(t,T_I)$ may be viewed as an exponential quadratic function of the state variables:

$$y_\gamma(t,T_I) = A_0(\gamma,t,T_I) + A_1(\gamma,t,T_I)Y(t) + \frac{1}{2}Y(t)' A_2(\gamma,t,T_I)Y(t) \tag{13}$$

$$B_\gamma(t,T_I) = \exp\left\{B_0(\gamma,t,T_I) + B_1(\gamma,t,T_I)'Y(t) + \frac{1}{2}Y(t)' B_2(\gamma,t,T_I)Y(t)\right\} \tag{14}$$

with the terminal condition $B_\gamma(T_I,T_I)=1$ implying that $B_0(\gamma,T_I,T_I) = B_1(\gamma,T_I,T_I) = B_2(\gamma,T_I,T_I) = 0$. $A_0(t,T_I)$, $A_1(t,T_I)$ and $A_2(t,T_I)$ are given in Appendix B.

$B_\gamma(t,T_I)$, which results from the agent's consumption-investment problem solution, is investor-specific, since it is a function of her (his) risk aversion coefficient and horizon. As its

expression, given in equation (12), is formally similar to that of a discount bond, it will be qualified to as the investor-specific discount bond. $B_\gamma(t,T_I)$ is stochastic because of the stochastic character of the prices of risk. For a more risk-averse investor than the logarithmic utility agent ($\gamma > 1$), $y_\gamma(t,T_I) > 0$, and $B_\gamma(t,T_I)$ is like a discounting factor. For example, when the investor's horizon increases, it tends very quickly to zero. Conversely, when (s)he is less risk-averse than the Bernoulli investor ($\gamma < 1$), $y_\gamma(t,T_I) < 0$, and $B_\gamma(t,T_I)$ is comparable to a compounding factor, in which case it tends to infinity for an increasing T_I. $y_\gamma(t,T_I)$ may be considered as a state variable incorporating the risk generated by the prices of risk and the yield curve. The investor-specific bond reveals that investors have a demand for bonds to satisfy their needs in terms of their horizon, their risk-preferences and their desire to hedge stochastic prices of risk. Under $P^{(\gamma,T_I)}$, the investor's optimization problem consists in calculating two bonds. The first is a discount bond (traded or synthetic) with a maturity date equal to the investor's horizon and is associated with interest rate risk (see Lioui and Poncet, 2001; Munk and Sorensen, 2004 and Detemple and Rindisbacher, 2009), while the second is an investor-specific discount "bond" (synthetic) reflecting time-variation in the prices of risk. Although $B_\gamma(t,T_I)$ is not a traded asset, it can be, however, manufactured, in a complete market, by existing securities. The second "bond" reveals that investors have a demand for bonds to satisfy their needs in terms of their horizon, their risk-preferences and their desire to hedge stochastic prices of risk. The investor's consumption-wealth problem reduces to the computation of the two bonds, which, by referring to the yield curve literature, is well-known and does not require a specific method.

At any date t, the wealth of the investor is composed of $\theta_S(t)$, $\theta_B(t)$ and $\theta_\beta(t)$ units of the spot commodity, the discount bonds and the riskless asset respectively, and the margin account:

$$W(t) = \theta_S(t)\left(S(t) + \int_0^t S(u)\delta(u)du \right) + \theta_B(t)B(t,T_B) + \theta_\beta(t)\beta(t) + M(t) \tag{15}$$

Applying Itô's lemma to the above expression and by the self-financing property, the dynamics of the unconstrained investor's wealth may be written:

$$\frac{dW(t)}{W(t)} = \left[r(t) + \pi(t)'\sigma\lambda(t) - \frac{c(t)}{W(t)} \right] dt + \pi(t)'\sigma dz(t) \tag{16}$$

with the initial condition *W(0)* and $\pi(t)' = \left[\pi_S(t) \quad \pi_B(t) \quad \pi_H(t)\right]$. $\pi_S(t) \equiv \frac{\theta_S(t)S(t)}{W(t)}$, $\pi_B(t) \equiv \frac{\theta_B(t)B(t,T_B)}{W(t)}$ and $\pi_H(t,T_H) \equiv \frac{\theta_H(t)H(t,T_H)}{W(t)}$ denote the proportions of the total wealth invested in the commodity, the discount bond and the futures contract respectively. In the sequel of the paper, we shall distinguish the speculative proportions, $\pi^{MV}(t)$, from the hedging proportions, $\pi^{HIR}(t)$, related to the discount bond $B(t,T_I)$ and those, $\pi^{HMPR}(t)$, related to the investor-specific bond, $B_\gamma(t,T_I)$. In order to optimally determine these proportions, the unconstrained investor solves the static program (8). The result obtained is presented in the following proposition.

Proposition 1. *Given the economic framework described above, the optimal demand for risky assets by the unconstrained investor is given by*:

$$\pi(t) = \frac{1}{\gamma}\Sigma^{-1}\sigma\lambda(t) + \left(1 - \frac{1}{\gamma}\right)\Sigma^{-1}\sigma\left[a^{\frac{1}{\gamma}} \int_t^{T_I} e^{-\frac{\eta}{\gamma}(s-t)} \frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)} \sigma_\varphi(\gamma,t,s)ds + (1-a)^{\frac{1}{\gamma}} e^{-\frac{\eta}{\gamma}(T_I-t)} \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)} \sigma_\varphi(\gamma,t,T_I) \right] \tag{17}$$

The optimal asset allocation may be decomposed in:

a) a traditional mean-variance component

$$\pi^{MV}(t) = \begin{pmatrix} \pi_S^{MV}(t) \\ \pi_B^{MV}(t) \\ \pi_H^{MV}(t) \end{pmatrix} = \frac{1}{\gamma}\Sigma^{-1}\sigma\lambda(t) = \frac{1}{\gamma}\Sigma^{-1}\begin{bmatrix} \mu_S(t) - r(t) \\ \mu_B(t) - r(t) \\ \mu_H(t) \end{bmatrix} \tag{18}$$

b) a hedging component related to the stochastic fluctuations in the instantaneous return of a discount bond with maturity date equal to the investor's horizon

$$\pi^{HIR}(t) = \begin{pmatrix} \pi_S^{HIR}(t) \\ \pi_B^{HIR}(t) \\ \pi_H^{HIR}(t) \end{pmatrix} = \left(1 - \frac{1}{\gamma}\right)\Sigma^{-1}\sigma\left[a^{\frac{1}{\gamma}} \int_t^{T_I} e^{-\frac{\eta}{\gamma}(s-t)} \frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)} \sigma_B(t,s)ds + (1-a)^{\frac{1}{\gamma}} e^{-\frac{\eta}{\gamma}(T_I-t)} \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)} \sigma_B(t,T_I) \right] \tag{19}$$

c) a hedging component related to the stochastic fluctuations in the instantaneous return of the investor-specific discount bond with maturity date equal to the investor's horizon

$$\pi^{HMPR}(t)=\begin{pmatrix}\pi_S^{HMPR}(t)\\ \pi_B^{HMPR}(t)\\ \pi_H^{HMPR}(t)\end{pmatrix}=\Sigma^{-1}\sigma\left[a^{\frac{1}{\gamma}}\int_t^{T_I}e^{-\frac{\eta}{\gamma}(s-t)}\frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)}\sigma_{B_\gamma}(t,s)ds+(1-a)^{\frac{1}{\gamma}}e^{-\frac{\eta}{\gamma}(T_I-t)}\frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\sigma_{B_\gamma}(t,T_I)\right] \tag{20}$$

where, $\sigma_\varphi(\gamma,t,T_I)=\left(1-\frac{1}{\gamma}\right)\sigma_B(t,T_I)+\sigma_{B_\gamma}(t,T_I)$, $\sigma_{B_\gamma}(t,T_I)=\sigma_Y^{'}\left[B_1(\gamma,t,T_I)+B_2(\gamma,t,T_I)Y(t)\right]$, $\Sigma=\sigma\sigma^{'}$ and $\Sigma_Y=\sigma_Y\sigma_Y^{'}$. $B_1(\gamma,t,T_I)$ *and* $B_2(\gamma,t,T_I)$ *are solutions to the following ordinary differential equations (ODEs):*

$$B_{2t}(\gamma,t,T_I)-B_2(\gamma,t,T_I)\overline{\mu}_{Y\gamma}-\overline{\mu}_{Y\gamma}^{'}B_2(\gamma,t,T_I)+B_2(\gamma,t,T_I)\Sigma_Y B_2(\gamma,t,T_I)-A_2(\gamma,t,T_I)=0$$
$$B_{1t}(\gamma,t,T_I)-\overline{\mu}_{Y\gamma}^{'}B_1(\gamma,t,T_I)+\overline{\mu}_\gamma(t)^{'}B_2(\gamma,t,T_I)+B_2(\gamma,t,T_I)\Sigma_Y B_1(\gamma,t,T_I)-A_1(\gamma,t,T_I)=0$$

with the terminal condition $B_1(\gamma,T_I,T_I)=B_2(\gamma,T_I,T_I)=0$. $B_{1t}(\gamma,t,T_I)$ and $B_{2t}(\gamma,t,T_I)$ *are the first order derivatives with respect to t. The constant and deterministic functions* $\overline{\mu}_{Y\gamma}$ *and* $\overline{\mu}_\gamma(t)$ *are given in Appendix B.*

Proof. See Appendix B.

Similar results can be found in other papers when. The main important difference is that these results are expressed in terms of the two discount bonds. As shown in Proposition 1, the optimal demand for risky assets (equation 17) can be decomposed into two parts. The first one is the traditional mean-variance speculative portfolio proportional to the investor's risk tolerance (Proposition 2 below is dedicated to this term), whereas the second part is a hedge portfolio (see Proposition 3 below). This portfolio, which corresponds to the second component of the right hand side of equation 17, serves as a hedge against the stochastic behavior of the instantaneous returns of $\Phi(\gamma,t,T_I)$ and $\Theta(t,T_I)$. Under the CRRA-forward measure, the investor does not hedge against changes in each and every state variable composing the investment opportunity set, but rather (s)he hedges against unfavorable shifts in $\Phi(\gamma,t,T_I)$ and $\Theta(t,T_I)$, which encompass all the uncertainty in the investment opportunity set.

$\Phi(\gamma,t,T_I)$ and $\Theta(t,T_I)$ depend on the two bonds, $B(t,T_I)$ and $B_\gamma(t,T_I)$. These two bonds represent, under the CRRA-forward measure, the two sources of risk in the economy. As a consequence, the hedging demand can be split in two components. The first one reveals how the investor should optimally hedge against random fluctuations in the instantaneous return of the discount

bond $B(t,T_I)$. The second ingredient hedges against the risk generated by the investor-specific discount bond and arises because the prices of risk are stochastic. Notice that if market prices of risk were assumed to be constant or deterministic, then only the discount bond $B(t,T_I)$ risk would be likely to be hedged by the investor. Although this term is similar in essence to that of other articles, since it captures the risk associated with the prices of risk, it is expressed in a different manner in our paper. In Detemple and Rindisbacher (2009) model, which is more general than ours, this component was couched in terms of the density of the forward measure, while our formula (20) highlights the role played by the investor-specific bond $B_\gamma(t,T_I)$. In Lioui and Poncet (2001) paper this addend depends on an unknown volatility function of the forward measure density, which was left unspecified. In contrast, we explicitly specify the volatility $\sigma_{B_\gamma}(t,T_I)$. These differences come from the change of the probability measure operated: on the one hand, the forward measure in the case of these papers, and, on the other hand, the CRRA-forward measure in our case.

The next propositions are devoted to a thorough study of the speculative and hedging terms. They try to elucidate the consequences on these terms of the stochastic opportunity set, especially the stochastic convenience yield, to highlight the role played by the traded primitive assets and the futures contract as hedging instruments, and, for practical considerations, to implement these terms. Moreover, Proposition 1 does not allow an investor to compute the optimal proportions for each risky asset. In this respect, we extend the previous literature (see also Adler and Detemple, 1988a, b) by deriving individual weighs for each asset.

To achieve these goals, two assets may be introduced into our analysis whose prices are denoted $B_u(t,T_B)$ and $H_v(t,T_H)$. These assets are assumed to be cash assets, i.e., they are not marked to market, and can be duplicated by a portfolio of four assets, namely the riskless asset, the discount bond with maturity T_B, the spot commodity and the futures contracts. They reflect orthogonal risks. The first asset is associated with the orthogonal risk of the interest rate, while the second one is linked to that of the convenience yield. Note that the existing spot commodity spans the risk of $z_S(t)$.

$$\frac{dB_u(t,T_B)}{B_u(t,T_B)} = r(t)dt - \sigma_{Bu}(t,T_B)' dz(t)$$

$$\frac{dH_v(t,T_H)}{H_v(t,T_H)} = r(t)dt + \bar{\sigma}_{Hv}(t,T_H)' dz(t)$$

with initial conditions $B_u(0,T_B)$ and $H_u(0,T_H)$ and where $\sigma_{Bu}(t,T_B)' = [0 \quad \rho_{ur}\sigma(t,T_B) \quad 0]$ and $\bar{\sigma}_{Hv}(t,T_H)' = [0 \quad 0 \quad -\sigma_{Hv}(t,T_H)]$.

Equation (18) may further be manipulated to obtain more insightful expressions by introducing the two synthetic assets into our analysis. This leads to the following proposition.

Proposition 2. *The optimal mean-variance proportions can be couched in a recursive way:*

$$\pi_H^{MV}(t) = \frac{1}{\gamma}\frac{\mu_{Hv}(t,T_H) - r(t)}{\sigma_{Hv}^2(t,T_H)} \tag{21}$$

$$\pi_B^{MV}(t) = \frac{1}{\gamma}\left[\frac{\mu_{Bu}(t,T_B) - r(t)}{\sigma_{Bu}(t,T_B)'\sigma_{Bu}(t,T_B)} - \frac{\Sigma_{HB_u}(t,T_H,T_B)}{\sigma_{Bu}(t,T_B)'\sigma_{Bu}(t,T_B)}\pi_H^{MV}(t)\right] \tag{22}$$

$$\pi_S^{MV}(t) = \frac{1}{\gamma}\left[\frac{\mu_S(t) - r(t)}{\sigma_S^2} - \frac{\Sigma_{SB}(t,T_B)}{\sigma_S^2}\pi_B^{MV}(t) - \frac{\Sigma_{HS}(t,T_H)}{\sigma_S^2}\pi_H^{MV}(t)\right] \tag{23}$$

Proof. See Appendix C.

This formulation is useful for computational purposes since speculative demands are expressed in terms of excess returns, volatilities and covariances, and they are calculated in a recursive way: the speculative demand of futures contracts is first derived, which allows one then to determine that of the discount bond and finally the proportion of the spot commodity can be obtained as a function of the other two demands.

The investor's speculative demand consists of a fund including an element specific to the futures contract and a component proper to the two primitive risky assets. This decomposition sheds light on the role played by the orthogonal risks captured by the two replicable assets and the spot commodity. The speculative demand for the futures contract depends on the excess return and the variance of the synthetic asset, $H_v(t,T_H)$. It reflects the investor's anticipations about the orthogonal source of uncertainty of the convenience yield. The futures contract is thus the sole asset that will be used by the investor to directly form her (his) expectations about the future evolution of the convenience yield. It follows that the mean-variance portfolio for futures contracts will be the only

demand depending uniquely on the price risk of the convenience yield. The speculative demand for the discount bond is a function of the excess return and the variance of the synthetic asset, $B_u(t,T_B)$, which spans the orthogonal risk of the interest rate. Because of the correlation of the futures contract with the short rate, $\pi_B^{MV}(t)$ is, however, modified by a second term. This additional term involves the mean-variance portfolio for futures contracts weighted by the usual covariance/variance ratio $\frac{\Sigma_{HB_u}(t,T_H,T_B)}{\sigma_{Bu}(t,T_B)'\sigma_{Bu}(t,T_B)}$. A similar argument applies to the speculative demand for commodities. The excess return of the spot commodity divided by its variance, spanning the orthogonal risk of the commodity, is now adjusted by two terms since the spot commodity is correlated with both the futures contract and the discount bond.

The interaction between the three components of the investor's speculative demand can be examined through the covariances between the assets. On the one hand, since $\Sigma_{SB}(t,T_B)$ is supposed to take low real values, it has a weak impact on the investor's speculative position on the spot commodity. As expected, unlike $\pi_S(t)$, the proportion invested in the discount bond is strongly influenced by $\Sigma_{HB_u}(t,T_H,T_B)$, and therefore by the speculative demand of the futures contracts. On the other hand, as the spot commodity and the futures contract are highly positively correlated, the speculative proportion of the commodity will be largely driven by that of the futures contract. Thus the parameters of the convenience yield and its price of risk are of paramount importance in determining the positions of the spot commodity and the discount bond.

Proposition 3. a) *The optimal hedging proportions spawned by changes in the discount bond* $B(t,T_I)$ *(interest rate risk) write:*

$$\pi_H^{HIR}(t)=\pi_S^{HIR}(t)=0$$

$$\pi_B^{HIR}(t)=\left(1-\frac{1}{\gamma}\right)\left[a^{\frac{1}{\gamma}}\int_t^{T_I} e^{-\frac{\eta}{\gamma}(s-t)}\frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)}\frac{\sigma(t,s)}{\sigma(t,T_B)}ds+(1-a)^{\frac{1}{\gamma}}e^{-\frac{\eta}{\gamma}(T_I-t)}\frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\frac{\sigma(t,T_I)}{\sigma(t,T_B)}\right] \tag{24}$$

b) The optimal hedging proportions generated by the investor-specific bond (market prices of risk) may be expressed in a recursive way for each risky asset:

$$\pi_H^{HMPR}(t) = -\frac{1}{\sigma_{Hv}(t,T_H)}\left[a^{\frac{1}{\gamma}}\int_t^{T_I} e^{-\frac{\eta}{\gamma}(s-t)}\frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)}\sigma_{B_\gamma v}(t,s)ds + (1-a)^{\frac{1}{\gamma}} e^{-\frac{\eta}{\gamma}(T_I-t)}\frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\sigma_{B_\gamma v}(t,T_I)\right] \tag{25'}$$

$$\begin{aligned}\pi_B^{HMPR}(t) = \frac{1}{\rho_{ur}\sigma(t,T_B)}&\left[a^{\frac{1}{\gamma}}\int_t^{T_I} e^{-\frac{\eta}{\gamma}(s-t)}\frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)}\sigma_{B_\gamma u}(t,s)ds + (1-a)^{\frac{1}{\gamma}} e^{-\frac{\eta}{\gamma}(T_I-t)}\frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\sigma_{B_\gamma u}(t,T_I)\right] \\ &-\frac{\Sigma_{HB_u}(t)}{\sigma_{Bu}(t,T_I)'\sigma_{Bu}(t,T_I)}\pi_H^{HMPR}(t)\end{aligned} \tag{25''}$$

$$\begin{aligned}\pi_S^{HMPR}(t) = \frac{1}{\sigma_S}&\left[a^{\frac{1}{\gamma}}\int_t^{T_I} e^{-\frac{\eta}{\gamma}(s-t)}\frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)}\sigma_{B_\gamma S}(t,s)ds + (1-a)^{\frac{1}{\gamma}} e^{-\frac{\eta}{\gamma}(T_I-t)}\frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\sigma_{B_\gamma S}(t,T_I)\right] \\ &-\frac{\Sigma_{SB}(t)}{\sigma_S^{\ 2}}\pi_B^{HMPR}(t) - \frac{\Sigma_{HS}(t)}{\sigma_S^{\ 2}}\pi_H^{HMPR}(t)\end{aligned} \tag{25'''}$$

c) The optimal hedging proportions generated by the investor-specific bond (market prices of risk) may be decomposed in the following manner for each and every state variable:

$$\pi^{HMPR}(t) = \pi^{HMPR_X}(t) + \pi^{HMPR_r}(t) + \pi^{HMPR_\delta}(t) \tag{26}$$

$$\begin{aligned}\pi^{HMPR_i}(t) = \Sigma^{-1}\sigma\overline{\sigma}_i&\left[a^{\frac{1}{\gamma}}\int_t^{T_I} e^{-\frac{\eta}{\gamma}(s-t)}\left[I_l^{'}B_1(t,s) + I_l^{'}B_2(t,s)Y(t)\right]\frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)}ds\right. \\ &\left.+(1-a)^{\frac{1}{\gamma}} e^{-\frac{\eta}{\gamma}(T_I-t)}\left[I_l^{'}B_1(t,T_I) + I_l^{'}B_2(t,T_I)Y(t)\right]\frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\right]\end{aligned} \tag{27}$$

$$\pi^{HMPR_i}(t) = \Sigma^{-1}\sigma\overline{\sigma}_i\left[a^{\frac{1}{\gamma}}\int_t^{T_I} e^{-\frac{\eta}{\gamma}(s-t)}\Psi_{\gamma_i}(t,s)\frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)}ds + (1-a)^{\frac{1}{\gamma}} e^{-\frac{\eta}{\gamma}(T_I-t)}\Psi_{\gamma_i}(t,T_I)\frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\right] \tag{27'}$$

where $i \in \{X(t), r(t), \delta(t)\}$, *I is a 3-dimensional identity matrix and* I_l, *l = 1, 2, 3, represent its columns.* $\sigma_{B_\gamma}(t,T_I) = \left[\sigma_{B_\gamma S}(t,T_I) \quad \sigma_{B_\gamma u}(t,T_I) \quad \sigma_{B_\gamma v}(t,T_I)\right]'$, $\Psi_{\gamma_i}(t,T_I) = \frac{B_{\gamma_i}(t,T_I)}{B_\gamma(t,T_I)}$ *and* $B_{\gamma_i}(t,T_I)$ *stands for the first order derivative of* $B_\gamma(t,T_I)$ *with respect to each state variable.*

Proof. See Appendix D.

Proposition 3 above has the traditional Merton-Breeden flavor as it disentangles the hedging proportions for each and every state variable. Indeed, the change of measure described above allows us providing a decomposition of the hedging element in terms of two discount bonds instead of an arbitrary finite number of Merton-Breeden terms equal to that of the state variables. These two bonds embed the risk associated with the short rate and the prices of risk respectively, the latter being

functions of the state variables. However, in order to investigate both the role of the risky assets and the impact of the state variables on the optimal hedging portfolio, that of the convenience yield especially, we further decompose the hedging components for each variable. In sharp contrast to the classical Merton-Breeden elements, our optimal hedging proportions inherit the advantages of Proposition 1, in that they depend on the features of the two bonds above mentioned.

According to Proposition 3, the hedging demand for the discount bond (equation 24) is the only one including a term that hedges the risk due to the stochastic nature of the interest rate through $B(t,T_I)$. This component is proportional to the ratio of the volatilities of the bonds with maturities respectively equal to T_I and T_B. When the two maturities coincide, this ratio is equal to one, and the hedging demand is merely a function of the two bonds $B(t,T_I)$ and $B_\gamma(t,T_I)$. This is also the sole ingredient in the agent's optimal demand evolving deterministically over time. This feature is quite general, in the sense that it is not related to the Gaussian character of the short rate. Insofar as the variance of the interest rate is proportional to its level this characteristic remains valid. This would be the case, for instance, if the short-rate followed a square-root process.

Parts b) and c) indicate that the hedging term that stems from the stochastic character of the investor-specific bond may admit two different decompositions pursuing two different objectives. The first, in the spirit of the speculative components (Proposition 2), expresses the hedging terms in a recursive way for each risky security. Furthermore, the covariances between the state variables and the assets which, in conjunction with the partial derivatives of $B_\gamma(t,T_I)$, determine the sign of the hedging demands, appear in a simple way facilitating the use of the above expressions.

The futures contract serves to hedge the orthogonal risk of the convenience yield, while the discount bond and the spot commodity are employed to hedge those of the short rate and the log of the spot commodity respectively. Correlations between the assets imply that $\pi_B^{HMPR}(t)$ and $\pi_S^{HMPR}(t)$ are appropriately adjusted to take into account for these correlations. Once again, similarly to the speculative elements, the parameters of the convenience yield have a heavy impact on the hedging terms of the spot commodity and the bond. The proportions obtained in Proposition 2 differ markedly from those of Breeden (1984), who, in his study, considers futures contracts with an instantaneous

maturity perfectly correlated with the state variables. This particular definition of futures contracts implies that the demand for the primitive assets that serve to hedge state variables disappears. In contrast, our investor elaborates her (his) strategy by including the primitive assets in order to hedge against the risk of the state variables.

The second decomposition, given in expression (26), separates the hedging addend into three components; one for each and every state variable. In particular, introducing a stochastic convenience yield into the economy results in the presence of a hedging demand, $\pi^{HMPR_\delta}(t)$, specific to this yield, that cannot be neglected. This equation makes it possible to disentangle the hedging element related to each state variable from those associated with the other variables. As a consequence, our model has the ability to exactly measure the impact of these hedging terms on the investor's optimal demand. It allows one to assess the weight of each state variable in the hedging terms stemming from the investor discount bond and therefore to assess the relevance as well as the importance of the state variables included in the investment opportunity set when the investor's objective is to implement hedging strategies. Actually, given the nature of the underlying commodity, some factors may have a strong or a negligible effect on these hedging elements implying that these factors may have or have not to be included in the opportunity set. Thus, this last may varied according to the nature of the spot commodity to be hedged. Since the hedging components are affine functions, the investor has, in addition, the possibility to separate the impact of the state variables on these components and to better understand the overall behavior of his (her) optimal demand.

In the light of expression (27'), this decomposition appears in a natural way and admits an economic interpretation. The investor seeks an insurance against the random shifts in the price of the investor-specific bond. As discussed above, $B_\gamma(t,T_I)$ incorporates the prices of risk through $y_\gamma(t)$, which involves the state variables. The hedging demands $\pi^{HMPR_i}(t)$ depend on the ratios $\Psi_{\gamma_i}(t,T_I)$. Since $B_\gamma(t,T_I)$ is analogous to a discount bond, these ratios determine its sensitivity on the three state variables (see also Wachter, 2002). In other words, each $\Psi_{\gamma_i}(t,T_I)$ assesses the sensitivity of the hedging demands to changes in $B_\gamma(t,T_I)$ resulting from a change in the state variables.

4. An illustrative example

To get more insights on the impact of the parameters on the model, various simulations are represented in figures 1 to 12. We simulate the reaction of the speculative and the hedging demands to the investor's horizon as well as to the state variables evolution. Table 1 summarizes the values of the parameters used in our simulations.

[INSERT TABLE 1 ABOUT HERE]

The parameter values are partly inspired from Schwartz's (1997) and Casassus and Collin-Dufresne's (2005) papers. They are chosen in order to account for some features characterizing commodities. Commodity futures prices are frequently below the current spot price exhibiting backwardation (see Litzenberger and Rabinowitz, 1995), which is equivalent to a positive risk premium and implies a positive convenience yield. Commodity spot prices and convenience yields follow mean-reverting processes (e.g. Bessembinder et al., 1995), as well as the short rate, so that $\alpha > 0$ and $k > 0$. The constant components of the prices of risk are supposed to be positive, while $\lambda_{XX} < 0, \lambda_{X\delta} < 0,$ and $\lambda_{rr} < 0$ inducing also mean-reversion in prices of risk and strengthening that of the state variables (see Cassasus and Collin-Dufresne, 2005). The convenience yield and the spot price are related through inventory decisions (e.g. Routledge et al., 2000). During periods of low inventories, the probability that shortages will occur is greater, and hence the spot price as well as the convenience yield should be high. Conversely, when inventories are abundant, the spot price and the convenience yield tend to be low. It follows that a positive correlation between the convenience yield and the spot price may be predicted. Frankel and Hardouvelis (1985) and Frankel (1986) argued that high real interest rates reduce commodity prices, and vice-versa. This should imply a negative correlation between, on the one hand, interest rates and, on the other hand, spot prices and convenience yields.

To analyze the impact of risk aversion, optimal demands are depicted for four degrees of relative risk aversion (RRA). The first one corresponds to the logarithmic function, $\gamma = 1$, in which case, the hedging terms vanish. The Bernoulli investor appears as the dividing line between hedging positions taken by investors who are more or less risk-averse. The second risk aversion parameter caracterizes an investor who is less risk-averse than the Bernoulli one. As pointed out by Kim and

Omberg (1996), the indirect utility function may explode for too low values when $\gamma < 1$. To avoid such a problem, we put 0.7. For a more risk-averse investor than the log-utility investor, we retain a value of $\gamma = 3$ for our simulations. Finally, according to Mehra and Prescott (1985) risk aversion should be much higher than one. To take into account this feature, we choose $\gamma = 6$.

When studying the optimal proportions as a function of the investment horizon, we let this horizon vary in the interval $T_I \in [0, 2]$, and we set the maturity of the futures contract and the bond such as $T_H = T_I + 1/12$ and $T_B = T_I + 5$ respectively. That is the futures contract and the discount bond expire one month and five years respectively after the end of the investor's horizon.

[INSERT FIGURES 1, 2, 3, 4 ABOUT HERE]

Figures 1 trough 4 picture the impact of the changes of the state variables on the speculative demand and of the γ parameter. We set the investor's horizon $T_I = 1$, while the other parameters values remain unchanged. As expected, the mean variance components are inversely related to γ and tend to zero as γ goes to infinity. Our numerical simulations show that the interest rate has a weak impact on the investor's demand, except for the proportions specific to this variable. We shall then omit the figures related to the short rate and shall mainly focus our analysis on the influence of the spot commodity price and the convenience yield, first, on the speculative demands, and, second, on the hedging terms.

To examine the role of the spot commodity, its price ranges from 80 dollars to 120 dollars. For low values of the spot price, the speculative component of the futures contract is positive and decreasing, while for high values it is negative and decreasing (see Figure 1). This result may essentially be explained by the price of risk associated with the orthogonal risk of the convenience yield, which is a function of those of the spot commodity and the convenience yield. The sign of $\pi_H^{MV}(t)$ depends on that of $\lambda_v(t)$ since the volatility $\sigma_{Hv}(t, T_H)$ is multiplied by minus - the convenience yield and the future price are negatively correlated. To understand this, formally, we have $\lambda_v(t) = \frac{1}{\rho_{v\delta}}\left[\frac{\rho_{Sr}\rho_{u\delta} - \rho_{S\delta}\rho_{ur}}{\rho_{ur}}\lambda_X(t) - \frac{\rho_{u\delta}}{\rho_{ur}}\lambda_r(t) + \lambda_\delta(t)\right]$. On the one hand, it depends negatively on the

price of risk related to the (log)spot price $(\rho_{Sr}\rho_{u\delta} - \rho_{S\delta}\rho_{ur} < 0)$ and positively to that of the convenience yield $(\rho_{v\delta} > 0)$. On the other hand, mean-reversion in the price of risk of the (log)spot price implies that as the latter raises the former declines. In overall, for low values of the (log)spot price, its price of risk outweighs that of the convenience yield so that $\lambda_v(t) < 0$, while for high values of the spot price the inverse holds. An inspection of Figure 1 shows that there exists a critical value of the spot price separating positive from negative speculative demands for futures contracts. In other words, the interaction between the prices of risk in conjunction with their mean-reverting behavior determines whether the speculator goes short or long.

Unlike $\pi_H^{MV}(t)$, $\pi_S^{MV}(t)$ is positive and monotonic increasing in the spot price (see Figure 2). When this last is low, a high, due to mean-reversion, instantaneous expected return of the spot commodity is negatively adjusted by both the speculative position in the futures contract $(\Sigma_{HS}(t,T_H) > 0)$ and in the bond $(\Sigma_{SB}(t,T_B) < 0)$. This results in a low speculative demand of the spot commodity and vice versa.

To underscore the importance of the convenience yield we let it vary between –5% and +15%. The speculative demand of the futures contract takes negative values and is an increasing function of the convenience yield (see Figure 3). Following the same reasoning as for the spot price, notice that the effect of the price of risk associated with the convenience yield dominates that of the spot commodity and the speculative demand is negative. However, as the convenience yield approaches to zero and becomes positive the difference between the two prices of risk lessens and tends to zero. For sufficiently high values of a positive convenience yield the speculative demand may be positive. The speculative proportion of the spot commodity is positive and decreasing in the convenience yield (see Figure 4). Indeed, the joint effect of mean-reversion in the expected return of the spot commodity and the behavior of the speculative demand of the futures contract lead to this result.

[INSERT FIGURES 5, 6, 7, 8, ABOUT HERE]

The reaction of the hedging addends to the changes of the state variables for different values of the γ parameter and of the optimal demands to the investor's horizon exhibit some features similar to

those reported, though in a different context, among others, by Kim and Omberg (1996), Campbell and Viceira (1999), Wachter (2002) and Sangvinatsos and Wachter (2005). These reactions are displayed in Figures 5 to 12. First, hedging positions as a function of the state variables may be short or long depending on the degree of risk-aversion. When $\gamma>1$, an investor facing unfavorable states of the world wishes to hedge against this adverse evolution of the investment opportunity set. By contrast, when $\gamma < 1$, an investor prefers to benefit from favorable states of the world and, hence, from a "good" investment opportunity set. An increase in the expected return is an improvement in the investment opportunity set, while a decrease in the expected return represents a worsening. As can be seen in Figure 5, for low prices of the spot commodity $\pi_H^{HMPR}(t)$ is positive (negative) for a less risk-averse (more risk-averse) agent than the Bernoulli one. However, as the spot price goes up, the sign of the investor's position reverses. A more risk-averse investor ($\gamma > 1$) desires to hedge the risk induced by time variation in the price of risk $\lambda_v(t)$, as a function of *X(t)*. As far as $\lambda_v(t)$ remains sufficiently negative (the return of $H_v(t,T_H)$ is positive), this investor has a positive hedging demand, $\pi_H^{HMPR}(t)$, while a more risk-tolerant ($\gamma<1$) agent has a negative one and vice versa. Figure 6 presents the impact of the convenience yield on $\pi_H^{HMPR}(t)$. The explanation is analogous to that of the impact of the spot commodity price. When the convenience yield is negative, $\lambda_v(t)>0$ but the expected return of $H_v(t,T_H)$ is negative. It follows that $\pi_H^{HMPR}(t)<0$ and $\pi_H^{HMPR}(t)>0$ for $\gamma>1$ and $\gamma<1$.

Second, in a particular region of the state variables, however, the hedging proportions behave in a different manner. The authors above mentioned provided an explanation in terms of mean-reversion in the state variables. A nonmyopic investor is aware not only about the current value of the prices of risk but also about their future value. As prices of risk are affine functions of the mean-reverting state variables, when they are negative, they are expected to pass through zero and to attain a positive value. An investor anticipating the future evolution of the price of risk, (s)he consequently modifies her (his) hedging demand. For example, Figure 5 shows the evolution of $\pi_H^{HMPR}(t)$ as a function of *X(t)*. For some values of the spot commodity price, when $\lambda_v(t)$ is negative but close to zero, the hedging demand is negative (positive) for a more risk-averse (risk-tolerant) investor. The mean-

reversion effect is less pronounced for the convenience yield and it can take place for very high positive values of $\delta(t)$, in which case the sign of the hedging proportions may be reversed (see Figure 6). Third, the hedging terms lie, for $\gamma = 6$, between those for $\gamma = 0$ and $\gamma = 3$ meaning that for more risk-averse agents than the logarithmic, beyond a certain level of the risk-aversion coefficient, the hedging demand decreases. This feature may be explained as follows. For high degrees of risk aversion, investors privilege less risky assets and limit their investment in risky assets whatever the level of the sate variables. Fourth, the hedging components stemming from the prices of risk associated with the futures contract and the spot commodity, in particular, are not monotonic in T_I (Figures 11 and 12): they are humped or inverted humped for a short horizon. This pattern may be explained by the mean-reverting character of the prices of risk when they attain values close to zero, in particular.

[INSERT FIGURES 9, 10, 11, 12 ABOUT HERE]

In addition, a clear distinction can be drawn between the mean-variance elements related to the interest rate (Figure no reproduced here) from those associated with the spot commodity and the futures contract (Figures 9 and 10). The latter are non-linear and sharply increase or decrease for a short horizon but they rapidly reach an asymptote for a longer term. This is due to the pattern of the synthetic assets price volatility, $\sigma_{Hv}(t,T_H)$: it flattens for a long horizon but is highly non-linear when the horizon shrinks. In contrast, the terms relative to the short rate are almost linear. Indeed, as the correlation between the interest rate and both the convenience yield and the commodity is low, these terms are essentially driven by the volatility of the bond. The latter slowly varies with the horizon, and, as a consequence, the demand for the bond.

4. Concluding remarks

In this paper, optimal hedging decisions involving commodity futures contracts have been studied in a continuous-time environment (i) for an unconstrained investor with a CRRA utility function, (ii) when spot prices, interest rates and, especially, the convenience yield evolve randomly over time, and (iii) the market price of risk associated with the spot commodity is stochastic and an affine function of the state

variables. In this setting, by using a suitable CRRA specific change of a martingale measure, we derive the investor's optimal demand, which consists of a speculative component and two so called hedging terms. The first hedging component is associated with interest rates uncertainty. This term involves a discount bond with a maturity equal to the investor's investment horizon. The second one deserves greater attention because it has some interesting properties distinguishing our results from those of other papers. It involves an investor-specific bond which can be used to hedge against fluctuations in the stochastic market price of risk. This component is composed of three hedging terms associated with the three state variables. They underline the role played by the primitive assets and the futures contracts as hedging instruments against the orthogonal risk of the state variables, the convenience yield in particular. Both the speculative component and the hedging terms can be couched in a recursive way improving the tractability of the model. The main implication of these properties is that the investor can measure the effect of each state variable on her (his) optimal demand and decide which of those variables are effectively important when s(he) pursues a hedging objective. This can be done, however, by exploiting the characteristics of the two bonds, i.e. the term structure of interest rates and the price of risk, instead of those of each and every state variable.

The economic framework of this paper can be extended in several directions. First, the general setting may usefully be adapted to the investor's allocation problem in the case of stocks paying a dividend. Second, commodities markets are highly volatile and spot assets exhibit jumps (see, for instance, Hilliard and Reis, 1998; Yan, 2002). The effect of jumps on the optimal asset allocation with commodities remains an open question. Third, another observed characteristic distinguishing commodities from financial assets is that commodity prices exhibit seasonal patterns (see Richter and Sorensen, 2006). It would be of great interest to examine how seasonality modifies the investor's hedging behavior. Finally, it is now acknowledged in the relevant literature that the convenience yield is not observable: indeed, in a partially observable economy (see, for instance, Dothan and Feldman 1986; Detemple 1986; Gennotte 1986; Xia 2001) an agent can estimate one or more unobserved state variable(s) given information conveyed by past observations spawned by observable state variables via the continuous-time Kalman-Bucy filter. One important extension would therefore be to study how the incomplete information affects optimal asset allocation.

Appendix A. Change of martingale measure.

In this Appendix, we determine the martingale measure to calculate the optimal consumption and wealth for any diffusion processes followed by the state variables and the prices of risk.

From expressions (9) and (10), it can be seen that the computation of $c(t)^*$ and $W(t)^*$ involves that of $\left(\frac{G(t)}{G(T_I)}\right)^{1-\frac{1}{\gamma}}$. To make this calculation easier, this term may be rewritten:

$$\left(\frac{G(t)}{G(T_I)}\right)^{1-\frac{1}{\gamma}} = B(t,T_I)^{1-\frac{1}{\gamma}}\left(\frac{\frac{B(T_I,T_I)}{G(T_I)}}{\frac{B(t,T_I)}{G(t)}}\right)^{1-\frac{1}{\gamma}} = B(t,T_I)^{1-\frac{1}{\gamma}}\left(\frac{R(T_I,T_I)}{R(t,T_I)}\right)^{1-\frac{1}{\gamma}}$$

where $B(t,T_I) = B(0,T_I)\exp\left\{\int_0^t\left[r(u)+\lambda(u)'\sigma_B(u,T_I)-\frac{1}{2}\sigma_B(u,T_I)'\sigma_B(u,T_I)\right]du+\int_0^t\sigma_B(u,T_I)'dz(u)\right\}$

is the price of a discount bond of maturity T_I and $R(t,T_I) = B(t,T_I)/G(t)$ is the relative price of this discount bond with respect to the numéraire *G(t)*.

The optimal wealth may be rewritten in the following convenient way:

$$W(t)^* = \zeta^{-\frac{1}{\gamma}}G(t)^{\frac{1}{\gamma}}\left[\int_t^{T_I}B(t,s)^{1-\frac{1}{\gamma}}E_t\left[\left(\frac{R(s,s)}{R(t,s)}\right)^{1-\frac{1}{\gamma}}\right]ds + B(t,T_I)^{1-\frac{1}{\gamma}}E_t\left[\left(\frac{R(T_I,T_I)}{R(t,T_I)}\right)^{1-\frac{1}{\gamma}}\right]\right] \tag{A.1}$$

Note that $R(t,T_I)$ is a martingale under the probability measure *P*, which leads to an immediate calculation of $\Theta(t,T_I)$. In contrast, $R(t,T_I)^{1-\frac{1}{\gamma}}$, for $\gamma<\infty$, is not a martingale under *P*. To see this, applying Ito's lemma to $R(t,T_I)^{1-\frac{1}{\gamma}}$ gives:

$$\begin{aligned} R(t,T_I)^{1-\frac{1}{\gamma}} = B(0,T_I)^{1-\frac{1}{\gamma}}&\exp\left\{-\frac{\gamma-1}{2\gamma^2}\int_0^t\left\|\lambda(u)-\sigma_B(u,T_I)\right\|^2du\right\} \\ &\exp\left\{-\frac{(1-\gamma)^2}{2\gamma^2}\int_0^t\left\|\lambda(u)-\sigma_B(u,T_I)\right\|^2du-\frac{1-\gamma}{\gamma}\int_0^t\left[\lambda(u)-\sigma_B(u,T_I)\right]'dz(u)\right\} \end{aligned} \tag{A.2}$$

with initial condition $R(0,T_I)^{1-\frac{1}{\gamma}} = B(0,T_I)^{1-\frac{1}{\gamma}}$.

Following Geman et al. (1995), the Radon-Nikodym derivative, defining the probability measure $P^{(\gamma,T_I)}$ equivalent to P, is given by:

$$\xi_\gamma(t,T_I) \equiv \left.\frac{dP^{(\gamma,T_I)}}{dP}\right|_{F_t} = \exp\left\{-\frac{(\gamma-1)^2}{2\gamma^2}\int_0^t \left\|\lambda(u)-\sigma_B(t,T_I)\right\|^2 du - \frac{\gamma-1}{\gamma}\int_0^t \left[\lambda(u)-\sigma_B(u,T_I)\right]' dz(u)\right\}$$

We suggest the following numéraire $N(t,T_I)=\xi_\gamma(t,T_I)G(t)$, associated with $P^{(\gamma,T_I)}$, such that any financial asset divided by $N(t,T_I)$ is a martingale under this new measure. The dynamics of this numéraire are governed by:

$$\frac{dN(t,T_I)}{N(t,T_I)} = \left[r(t)+\lambda(t)'\left[\frac{1}{\gamma}\lambda(t)+\left(1-\frac{1}{\gamma}\right)\sigma_B(t,T_I)\right]'\right]dt + \left[\frac{1}{\gamma}\lambda(t)+\left(1-\frac{1}{\gamma}\right)\sigma_B(t,T_I)\right]' dz(t) \qquad \text{(A.3)}$$

with the initial condition $N(0,T_I)=1$.

From equation (A.2), it follows that:

$$E_t\left[\left(\frac{R(T_I,T_I)}{R(t,T_I)}\right)^{1-\frac{1}{\gamma}}\right] = E_t\left[\exp\left\{-\frac{\gamma-1}{2\gamma^2}\int_t^{T_I}\left\|\lambda(u)-\sigma_B(u,T_I)\right\|^2 du\right\}\right.$$

$$\left.\exp\left\{-\frac{(1-\gamma)^2}{2\gamma^2}\int_t^{T_I}\left\|\lambda(u)-\sigma_B(u,T_I)\right\|^2 du - \frac{1-\gamma}{\gamma}\int_t^{T_I}\left[\lambda(u)-\sigma_B(u,T_I)\right]' dz(u)\right\}\right]$$

We can use Bayes' rule to get:

$$E_t\left[\left(\frac{R(T_I,T_I)}{R(t,T_I)}\right)^{1-\frac{1}{\gamma}}\right] = E_t^{(P^{\gamma,T_I})}\left[\exp\left\{-\int_t^{T_I}\frac{\gamma-1}{2\gamma^2}\left\|\lambda(u)-\sigma_B(u,T_I)\right\|^2 du\right\}\right]$$

$$= E_t^{(P^{\gamma,T_I})}\left[\exp\left\{-\int_t^{T_I} y_\gamma(u)du\right\}\right] = B_\gamma(Y(t),t,T_I) \equiv B_\gamma(t,T_I)$$

The same procedure may be used to compute:

$$\int_t^{T_I} E_t\left[\left(\frac{R(s,s)}{R(t,s)}\right)^{1-\frac{1}{\gamma}}\right]ds = \int_t^{T_I} E_t^{(P^{\gamma,s})}\left[\exp\left\{-\int_t^{s}\frac{\gamma-1}{2\gamma^2}\left\|\lambda(u)-\sigma_B(u,s)\right\|^2 du\right\}\right]ds = \int_t^{T_I} B_\gamma(t,s)ds$$

$P^{(\gamma,T_I)}$ is the CRRA T_I-forward probability measure or the CRRA-forward measure. $\xi_\gamma(t,T_I)$ is the density of the CRRA-forward measure with respect to *P*. $\frac{1-\gamma}{\gamma}[\lambda(t)-\sigma_B(t,T_I)]$ represents the price of risk expressed under the new probability measure.

Appendix B. Proof of Proposition 1.

By using expression (4) and by operating the appropriate calculations $y_\gamma(t)$ can be expressed as a quadratic function of the state variables.

$$y_\gamma(t,T_I)=\frac{\gamma-1}{2\gamma^2}[\lambda(t)-\sigma_B(t,T_I)]'[\lambda(t)-\sigma_B(t,T_I)]=\frac{\gamma-1}{2\gamma^2}[\lambda_0-\sigma_B(t,T_I)+\lambda_Y Y(t)]'[\lambda_0-\sigma_B(t,T_I)+\lambda_Y Y(t)]$$

$$=\frac{\gamma-1}{2\gamma^2}\left\{\|\lambda_0-\sigma_B(t,T_I)\|^2+2[\lambda_0-\sigma_B(t,T_I)]'\lambda_Y Y(t)+Y(t)'\lambda_Y'\lambda_Y Y(t)\right\}$$

$$=A_0(\gamma,t,T_I)+A_1(\gamma,t,T_I)Y(t)+\frac{1}{2}Y(t)'A_2(\gamma,t,T_I)Y(t)$$

where

$$\lambda_0=\begin{pmatrix}\lambda_{X0}\\ -\frac{\rho_{Sr}}{\rho_{ur}}\lambda_{X0}+\frac{1}{\rho_{ur}}\lambda_{r0}\\ \frac{\rho_{Sr}\rho_{u\delta}-\rho_{S\delta}\rho_{ur}}{\rho_{ur}\rho_{v\delta}}\lambda_{X0}-\frac{\rho_{u\delta}}{\rho_{ur}\rho_{v\delta}}\lambda_{r0}+\frac{1}{\rho_{v\delta}}\lambda_{\delta 0}\end{pmatrix}\text{ and}$$

$$\lambda_Y=\begin{pmatrix}\lambda_{XX} & 0 & \lambda_{X\delta}\\ -\frac{\rho_{Sr}}{\rho_{ur}}\lambda_{XX} & \frac{1}{\rho_{ur}}\lambda_{rr} & -\frac{\rho_{Sr}}{\rho_{ur}}\lambda_{X\delta}\\ \frac{\rho_{Sr}\rho_{u\delta}-\rho_{S\delta}\rho_{ur}}{\rho_{ur}\rho_{v\delta}}\lambda_{XX} & -\frac{\rho_{u\delta}}{\rho_{ur}\rho_{v\delta}}\lambda_{rr} & \frac{\rho_{Sr}\rho_{u\delta}-\rho_{S\delta}\rho_{ur}}{\rho_{ur}\rho_{v\delta}}\lambda_{X\delta}+\frac{1}{\rho_{v\delta}}\lambda_{\delta\delta}\end{pmatrix}$$

The dynamics of the state variables under the probability $P^{(\gamma,T_I)}$ are given by:

$$dY(t)=[\bar{\mu}_\gamma(t)-\bar{\mu}_{Y,\gamma}Y(t)]dt+\sigma_Y' dz^{(\gamma,T_I)}(t)$$

with the initial condition *Y(0)* = *Y*. $\bar{\mu}(t)=\begin{bmatrix}\sigma_S\lambda_{X0}-\frac{1}{2}\sigma_S^2\\ \vartheta(t)\\ k\bar{\delta}\end{bmatrix}$, $\bar{\mu}_Y=\begin{bmatrix}-\sigma_S\lambda_{XX} & -1 & 1-\sigma_S\lambda_{X\delta}\\ 0 & \alpha-\sigma_r\lambda_{rr} & 0\\ 0 & 0 & k\end{bmatrix}$,

$\sigma_Y=\begin{pmatrix}\sigma_S & 0 & 0\\ \rho_{Sr}\sigma_r & \rho_{ur}\sigma_r & 0\\ \rho_{S\delta}\sigma_\delta & \rho_{u\delta}\sigma_\delta & \rho_{v\delta}\sigma_\delta\end{pmatrix}$, $\bar{\mu}_{Y\gamma}(t)=\bar{\mu}(t)-\left(1-\frac{1}{\gamma}\right)\sigma_Y'[\lambda_0-\sigma_B(t,T_I)]$, $\bar{\mu}_{Y\gamma}=\bar{\mu}_Y+\left(1-\frac{1}{\gamma}\right)\sigma_Y'\lambda_Y$, and,

by Girsanov's theorem, $dz^{(\gamma,T_I)}(t) = dz(t) + \left(1-\frac{1}{\gamma}\right)[\lambda(t) - \sigma_B(t,T_I)]dt$ is a Brownian motion under $P^{(\gamma,T_I)}$. Moreover, let $\bar{\sigma}_X' = [\sigma_S \quad 0 \quad 0]$, $\bar{\sigma}_r' = [\rho_{sr}\sigma_r \quad \rho_{ur}\sigma_r \quad 0]$ and $\bar{\sigma}_\delta' = [\rho_{S\delta}\sigma_\delta \quad \rho_{u\delta}\sigma_\delta \quad \rho_{v\delta}\sigma_\delta]$ denote the 3-dimensional diffusion vectors of the state variables.

Following the relevant literature, $B_\gamma(t,T_I)$ is an exponential quadratic function of the state variables given in equation (14). $B_0(\gamma,t,T_I), B_1(\gamma,t,T_I)$ and $B_2(\gamma,t,T_I)$ satisfy three ODEs, subject to the terminal conditions, which are now well-known.

By using Itô lemma, $B_\gamma(t,T_I)$ and $\varphi(\gamma,t,T_I)$ follow respectively the SDEs:

$$\frac{dB_\gamma(t,T_I)}{B_\gamma(t,T_I)} = \mu_\gamma(t,T_I)dt + \sigma_{B_\gamma}(t,T_I)' dz^{(\gamma,T_I)}(t)$$

$$\frac{d\varphi(\gamma,t,T_I)}{\varphi(\gamma,t,T_I)} = \mu_\varphi(t,T_I)dt + \sigma_\varphi(\gamma,t,T_I)' dz^{(\gamma,T_I)}(t)$$

where $\sigma_{B_\gamma}(t,T_I) = \sigma_Y' \frac{B_{\gamma_Y}(t,T_I)}{B_\gamma(t,T_I)} = \sigma_Y'[B_1(\gamma,t,T_I) + B_2(\gamma,t,T_I)Y(t)]$ and $\sigma_\phi(\gamma,t,T_I) = \left(1-\frac{1}{\gamma}\right)\sigma_B(t,T_I) + \sigma_{B_\gamma}(t,T_I)$.

Note that $\mu_\gamma(t,T_I)$ and $\mu_\phi(t,T_I)$ are irrelevant for our allocation problem and will not be specified. By using Leibniz type rule for stochastic integrals (see Munk and Sorensen, 2004), $\Phi(\gamma,t,T_I)$ and $\Theta(t,T_I)$ obey the following equations:

$$\begin{aligned}\frac{d\Phi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)} = \mu_\Phi(\gamma,t,T_I)dt + \Bigg[& a^{\frac{1}{\gamma}} \int_t^{T_I} e^{-\frac{\eta}{\gamma}(s-t)} \frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)} \sigma_\varphi(\gamma,t,s)ds \\ & + (1-a)^{\frac{1}{\gamma}} e^{-\frac{\eta}{\gamma}(T_I-t)} \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)} \sigma_\varphi(\gamma,t,T_I) \Bigg]' dz^{(\gamma,T_I)}(t)\end{aligned} \tag{B.2}$$

By using Itô's lemma, the instantaneous return of the optimal wealth (15) may be written:

$$\begin{aligned}\frac{dW(t)^*}{W(t)^*} = \mu_{W^*}(t)dt + \Bigg[& \frac{\lambda(t)}{\gamma} + a^{\frac{1}{\gamma}} \int_t^{T_I} e^{-\frac{\eta}{\gamma}(s-t)} \frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)} \sigma_\varphi(\gamma,t,s)ds \\ & + (1-a)^{\frac{1}{\gamma}} e^{-\frac{\eta}{\gamma}(T_I-t)} \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)} \sigma_\varphi(\gamma,t,T_I) \Bigg]' dz^{(\gamma,T_I)}\end{aligned} \tag{B.3}$$

Identifying the diffusion terms of the admissible wealth (16) and the optimum wealth (B.3) yields:

$$\sigma\sigma'\pi = \sigma\left[\frac{\lambda(t)}{\gamma} + a^{\frac{1}{\gamma}}\int_t^{T_I} e^{-\frac{\eta}{\gamma}(s-t)}\frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)}\sigma_\varphi(\gamma,t,s)ds + (1-a)^{\frac{1}{\gamma}} e^{-\frac{\eta}{\gamma}(T_I-t)}\frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\sigma_\varphi(\gamma,t,T_I)\right] \tag{B.4}$$

which leads to equation (17).

Parts a), b) and c) of Proposition 1 can directly be obtained from this equation.

Appendix C. Proof of Proposition 2.

The following matrix products give:

$$\Sigma^{-1}\sigma = \begin{pmatrix} \frac{1}{\sigma_S} & -\frac{\rho_{Sr}}{\rho_{ur}\sigma_S} & -\frac{\rho_{ur}\sigma_{HS}(t,T_H) - \rho_{Sr}\sigma_{Hu}(t,T_H)}{\rho_{ur}\sigma_S\sigma_{Hv}(t,T_H)} \\ 0 & \frac{1}{\rho_{ur}\sigma(t,T_B)} & -\frac{\sigma_{Hu}(t,T_H)}{\rho_{ur}\sigma(t,T_B)\sigma_{Hv}(t,T_H)} \\ 0 & 0 & \frac{1}{\sigma_{Hv}(t,T_H)} \end{pmatrix} \tag{C.1}$$

$$\Sigma^{-1}\sigma\lambda(t) = \begin{pmatrix} \frac{\lambda_s(t)}{\sigma_S} - \frac{\lambda_u(t)\rho_{Sr}}{\rho_{ur}\sigma_S} - \frac{\lambda_v(t)}{\sigma_S}\frac{\rho_{ur}\sigma_{HS}(t,T_H) - \rho_{Sr}\sigma_{Hu}(t,T_H)}{\rho_{ur}\sigma_{Hv}(t,T_H)} \\ \frac{\lambda_u(t)}{\rho_{ur}\sigma(t,T_B)} - \frac{\lambda_v(t)\sigma_{Hu}(t,T_H)}{\rho_{ur}\sigma(t,T_B)\sigma_{Hv}(t,T_H)} \\ \frac{\lambda_v(t)}{\sigma_{Hv}(t,T_H)} \end{pmatrix}$$

By using equation (18) and by rearranging terms, we obtain:

$$\pi_{\mathrm{H}}(\mathrm{t}) = \frac{\lambda_v(t)\sigma_{Hv}(t,T_H)}{\sigma_{Hv}^2(t,T_H)} = \frac{\mu_{Hv}(t,T_H) - r(t)}{\sigma_{Hv}^2(t,T_H)} \tag{C.2}$$

$$\pi_{\mathrm{B}}(\mathrm{t}) = \frac{\rho_{ur}\sigma(t,T_B)\lambda_u(t)}{\rho_{ur}^2\sigma(t,T_B)\sigma(t,T_B)'} - \frac{\rho_{ur}\sigma(t,T_B)\sigma_{Hu}(t,T_H)}{\rho_{ur}^2\sigma(t,T_B)\sigma(t,T_B)'}\frac{\lambda_v(t)\sigma_{Hv}(t,T_H)}{\sigma_{Hv}^2(t,T_H)} \tag{C.3}$$

$$\pi_{\mathrm{S}}(\mathrm{t}) = \frac{\lambda_{\mathrm{S}}(t)}{\sigma_S} - \frac{\rho_{uf}\sigma_S\sigma(t,T_B)}{\sigma_S^2}\pi_{\mathrm{B}}(\mathrm{t}) - \frac{\sigma_S\sigma_{HS}(t,T_H)}{\sigma_S^2}\pi_{\mathrm{H}}(\mathrm{t}) \tag{C.4}$$

By replacing the first two moments and the appropriate covariances of the synthetic assets into (C.3) and (C.4), equations (21), (22) and (23), in the main text, are obtained.

Appendix D. Proof of Proposition 3.

a) Expression (24) can easily be derived by operating the computation of $\Sigma^{-1}\sigma\sigma_B(t,T_I)$.

b) $\sigma_{B_\gamma}(t,T_I)$ may be written in the following manner:

$$\sigma_{B_\gamma}(t,T_I)=\left[\sigma_{B_\gamma S}(t,T_I)\quad \sigma_{B_\gamma u}(t,T_I)\quad \sigma_{B_\gamma v}(t,T_I)\right]$$

Equation (20) may thus be expressed as:

$$\pi^{HMPR}(t)=\Sigma^{-1}\sigma\begin{bmatrix} a^{\frac{1}{\gamma}}\int_t^{T_I} e^{-\frac{\eta}{\gamma}(s-t)}\frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)}\sigma_{B_\gamma S}(t,s)ds+(1-a)^{\frac{1}{\gamma}}e^{-\frac{\eta}{\gamma}(T_I-t)}\frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\sigma_{B_\gamma S}(t,T_I) \\ a^{\frac{1}{\gamma}}\int_t^{T_I} e^{-\frac{\eta}{\gamma}(s-t)}\frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)}\sigma_{B_\gamma u}(t,s)ds+(1-a)^{\frac{1}{\gamma}}e^{-\frac{\eta}{\gamma}(T_I-t)}\frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\sigma_{B_\gamma u}(t,T_I) \\ a^{\frac{1}{\gamma}}\int_t^{T_I} e^{-\frac{\eta}{\gamma}(s-t)}\frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)}\sigma_{B_\gamma v}(t,s)ds+(1-a)^{\frac{1}{\gamma}}e^{-\frac{\eta}{\gamma}(T_I-t)}\frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\sigma_{B_\gamma v}(t,T_I) \end{bmatrix}$$

Using (C.1) leads to equations (25).

c) Let I be a 3-dimensional identity matrix and I_1, I_2, I_3 be its columns. Then, we have

$$\begin{aligned}\sigma_{B_\gamma}(t,T_I)=\sigma_Y^{'}\left[B_1(\gamma,t,T_I)+B_2(\gamma,t,T_I)Y(t)\right]=\bar{\sigma}_X\left[I_1^{'}B_1(\gamma,t,T_I)+I_1^{'}B_2(\gamma,t,T_I)Y(t)\right]+\\ \bar{\sigma}_r\left[I_2^{'}B_1(\gamma,t,T_I)+I_2^{'}B_2(\gamma,t,T_I)Y(t)\right]+\bar{\sigma}_\delta\left[I_3^{'}B_1(\gamma,t,T_I)+I_3^{'}B_2(\gamma,t,T_I)Y(t)\right]\end{aligned} \tag{D.1}$$

Plugging the above expression into equation (20) and rearranging terms leads to equations (26) and (27).

The first order derivative of $B_\gamma(t,T_I)$ with respect to the state variables can be written: $B_{\gamma_i}(t,T_I)=B_\gamma(t,T_I)\left[I_l^{'}B_1(\gamma,t,T_I)+I_l^{'}B_2(\gamma,t,T_I)Y(t)\right],\ i\in\{X(t),r(t),\delta(t)\}$ and $l = 1, 2, 3$. Replacing these derivatives into (D.1) gives:

$$\sigma_{B_\gamma}(t,T_I)=\sigma_Y^{'}\left[B_1(\gamma,t,T_I)+B_2(\gamma,t,T_I)Y(t)\right]=\bar{\sigma}_X\frac{B_{\gamma_X}(t,T_I)}{B_\gamma(t,T_I)}+\bar{\sigma}_r\frac{B_{\gamma_r}(t,T_I)}{B_\gamma(t,T_I)}+\bar{\sigma}_\delta\frac{B_{\gamma_\delta}(t,T_I)}{B_\gamma(t,T_I)} \tag{D.2}$$

Expressions (D.2) and (24) allow one to establish equation (27').

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Table 1

Numerical values of the parameters used in the model

r	δ	X	α	κ	$\bar{\delta}$	σ_S	σ_r	σ_δ	$f(0,t)$
0.04	0.07	4.6	0.25	1.5	0.05	0.35	0.01	0.25	0.04
λ_{X0}	λ_{r0}	$\lambda_{\delta 0}$	λ_{XX}	$\lambda_{X\delta}$	λ_{rr}	$\lambda_{\delta\delta}$	ρ_{Sr}	$\rho_{S\delta}$	$\rho_{r\delta}$
3.9	0.04	0.5	-0.8	-1.5	-0.15	-1.7	-0.15	0.7	-0.1

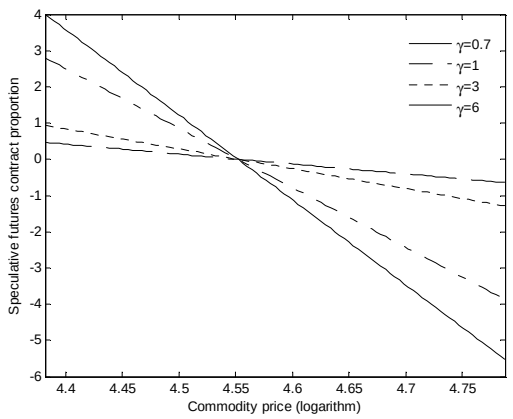

Fig. 1. *Speculative futures proportion varying with the commodity price.* This figure plots $\pi_H^{MV}(t)$ as a function of the (logarithm) commodity price ranging from \$80 to \$120 for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

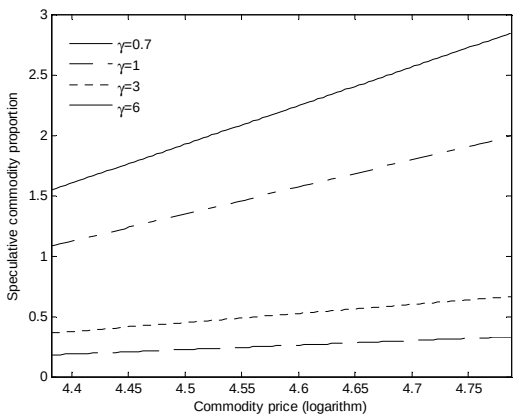

Fig. 2. *Speculative commodity proportion varying with the commodity price.* This figure plots $\pi_S^{MV}(t)$ as a function of the (logarithm) commodity price ranging from \$80 to \$120 for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

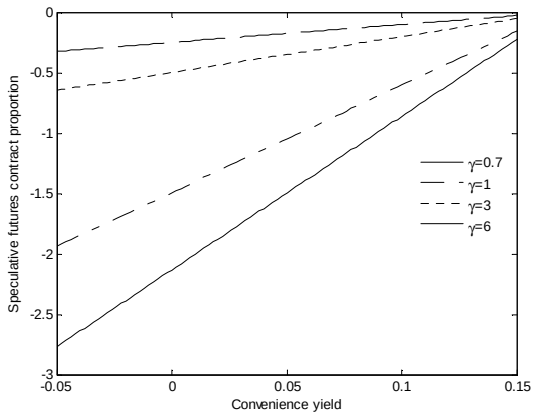

Fig. 3. *Speculative futures proportion varying with the convenience yield.* This figure plots $\pi_H^{MV}(t)$ as a function of the convenience yield ranging from -5% to 15% for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

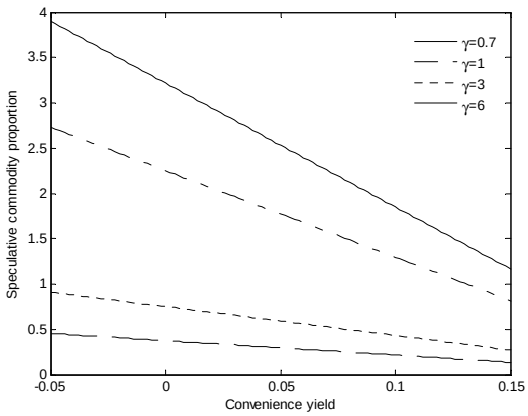

Fig.4 *Speculative commodity proportion varying with the convenience yield.* This figure plots $\pi_S^{MV}(t)$ as a function of the convenience yield ranging from -5% to 15% for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

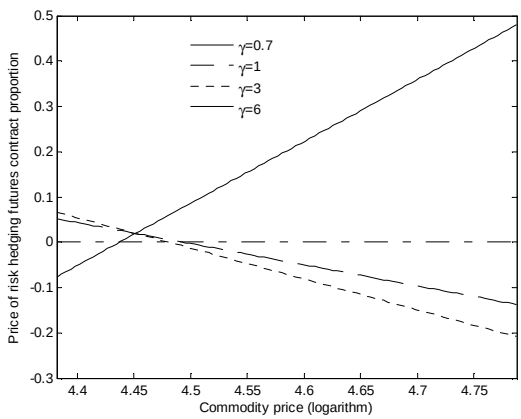

Fig. 5. Investor specific bond (p*rices of risk) hedging futures proportion varying with the commodity price.* This figure plots $\pi_H^{HMPR}(t)$ as a function of the (logarithm) commodity price ranging from \$80 to \$120 for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

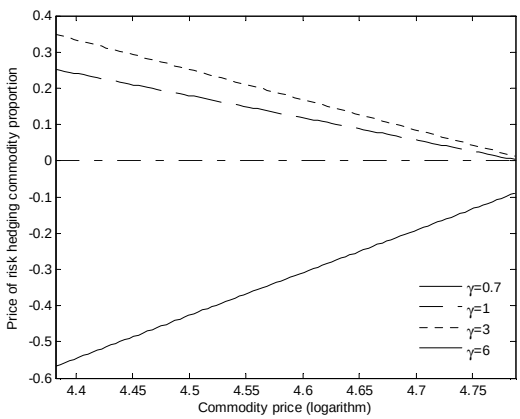

Fig. 6. Investor specific bond (p*rices of risk) hedging commodity proportion varying with the commodity price.* This figure plots $\pi_S^{HMPR}(t)$ as a function of the (logarithm) commodity price ranging from \$80 to \$120 for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

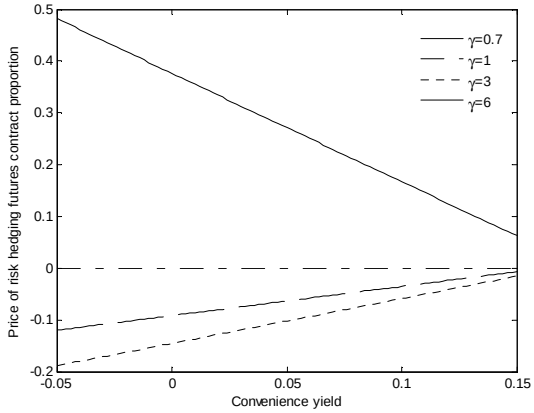

Fig. 7. Investor specific bond (p*rices of risk) hedging futures proportion varying with the convenience yield.* This figure plots $\pi_H^{HMPR}(t)$ as a function of the convenience yield ranging from -5% to 15% for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

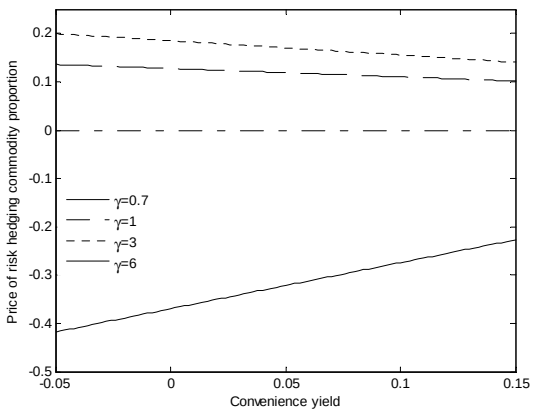

Fig. 8. Investor specific bond (p*rices of risk) hedging commodity proportion varying with the convenience yield.* This figure plots $\pi_S^{HMPR}(t)$ as a function of the convenience yield ranging from -5% to 15% for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

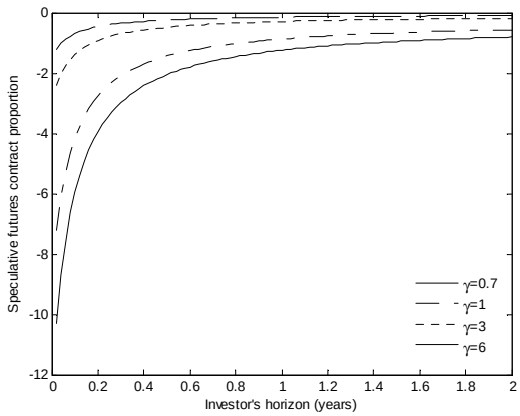

Fig. 9. *Speculative futures proportion varying with the investor's horizon.* This figure plots $\pi_H^{MV}(t)$ as a function of the investor horizon ranging from 0 to 2 years for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

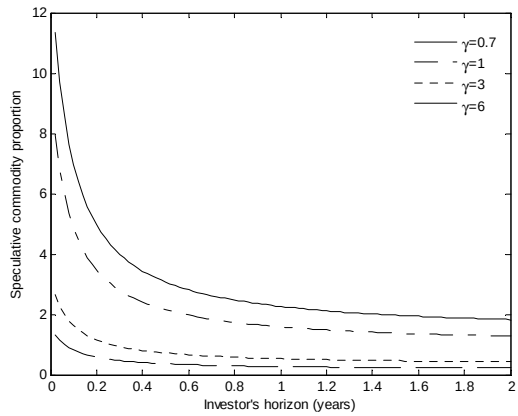

Fig. 10 *Speculative commodity proportion varying with the investor's horizon.* This figure plots $\pi_S^{MV}(t)$ as a function of the investor horizon ranging from 0 to 2 years for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

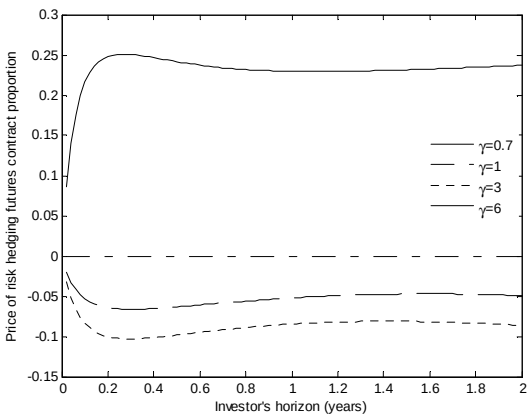

Fig. 11. Investor-specific bond (p*rices of risk) hedging futures proportion varying with the investor's horizon.* This figure plots $\pi_H^{HMPR}(t)$ as a function of the investor horizon ranging from 0 to 2 years for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

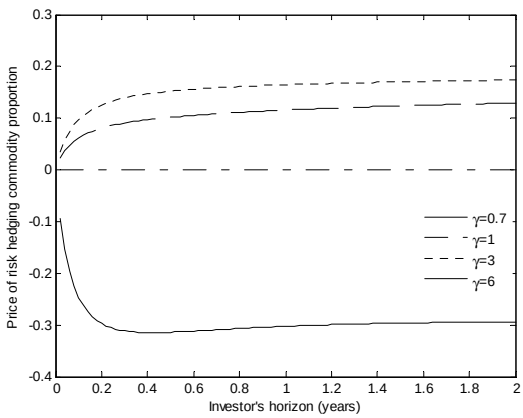

Fig. 12. Investor-specific bond (p*rices of risk) hedging commodity proportion varying with the investor's horizon.* This figure plots $\pi_S^{HMPR}(t)$ as a function of the investor horizon ranging from 0 to 2 years for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.