Equity-Index Options and CDOs, before and after the Subprime Crisis

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Abstract

Put options on equity indices and spreads on CDO-index tranches both reflect the downside risk on portfolios of firms. We test for the consistency of their behaviour before and after the subprime crisis, using weekly data from October 2006 to November 2007. To do this we develop an equity-implied copula model to price CDO tranches, in which the Gaussian distribution of the conventional copula is replaced by the left-skewed distribution from equity-index puts. We find that the new model is able to generate realistic spreads for CDO tranches without the need for an ad-hoc correlation skew (as used by traders to raise the frequency of default on more senior tranches). The model spreads on safer tranches of CDOs before the subprime crisis are significantly larger than those in the market, particularly for the senior (15-30) tranche for which model spreads are 12-18 basis points and market spreads are only 3-5 basis points. We conclude that information on downside risk available from the equity market was not being used correctly in setting spreads on CDO tranches before the subprime crisis.

JEL classification: G12, G13

Keywords: CDO, credit spread, index option, volatility smile, pricing kernel, copula

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1. Introduction

CDO tranches allow investors to take specific views on the shape of the credit-loss distribution on a diversified portfolio of bonds. Similarly, equity-index put options allow investors to take well-defined positions which depend on the downside returns of a diversified portfolio of equities. In both cases the factor generating returns is a portfolio of firms' values. If the prices of CDOs indicate that firms are at risk of defaulting in clusters, then the same clustered risk should be visible in the prices of equity-index put options.

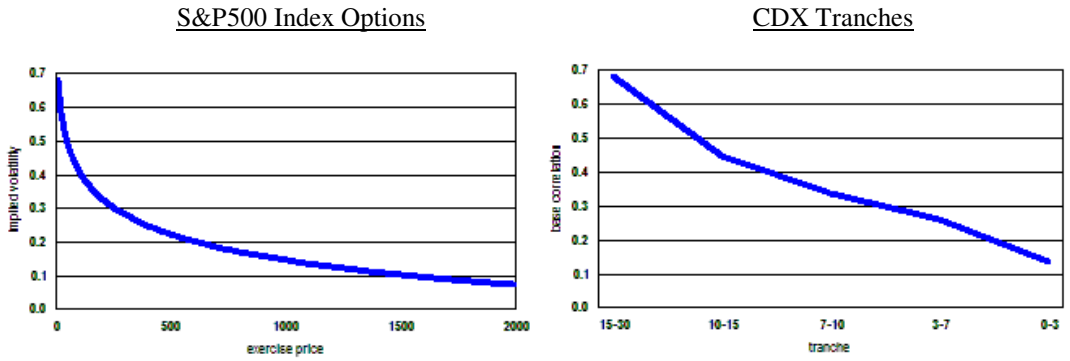
The conventional model for pricing put options is that of Black and Scholes (1973) and for pricing CDOs is the Gaussian copula (Li, 2000). So widespread is the use of these models that traders do not deal in terms of option prices or CDO spreads, but in terms of 'implied volatilities' and 'implied base correlations'. For an equity option, quoting the implied volatility (together with the levels of the exercise price and underlying stock price) is equivalent to quoting its premium with the Black-Scholes model. Similarly, for a CDO tranche, quoting the implied base correlation [1] (together with the spread level of the reference portfolio and the detachment level of the tranche) is equivalent to quoting the spread of that tranche with the standard one-factor Gaussian copula model.

It is well known that the implied volatilities of equity-index options become larger as the exercise price falls; similarly, base correlations of CDO tranches become larger as we move from junior to senior tranches. These empirical regularities are known as the 'volatility sneer' and 'base-correlation skew' respectively. Figure 1 gives examples of the implied-volatility sneer and the base-correlation skew on 27th October

[1] Base correlation is the correlation of default across bonds in a CDO, as implied by a tranche which is exposed to all losses above zero. For example, it might be quoted for a tranche exposed to losses of 0-3%, or quoted for a tranche exposed to losses of 0-30%. A 0-3% tranche would be a conventional equity tranche, with a market price, but a 0-30% tranche does not exist and its price has to be constructed for the base correlation by summing the prices of the intervening tranches (such as 0-3%, 3-7%, 7-10%, 10-15% and 15-30%). Base correlation can be contrasted with compound correlation, which is the simple correlation of defaults implied by a tranche. Compound correlations are not necessarily unique and that is why base correlations are usually preferred. For a discussion, see Willemann (2005).

2006.[2] As shown in the figure, the deeper out-of-the-money equity-index puts are priced at higher implied volatilities, which indicates that there is a much fatter downside (left) tail of the asset-return distribution than would arise with a normal distribution. Similarly, the more-senior tranches of the CDO are priced at higher levels of base correlation, which implies a fatter downside tail of the default-loss distribution of the portfolio than from a normal (Gaussian) distribution.

Figure 1 Equity-Volatility Sneer and Base-Correlation Skew on 27th October 2006.



The sneer and skew indicate that the Black/Scholes model for index options and the Gaussian-copula model for CDO tranches are both mis-specified. Both models assume that the relevant factor returns are normally distributed. Distributions of realised returns on stocks and corporate bonds are close to being normal, so the non-normality reflected in volatility sneers and correlation skews is not present in the physical domain but only in the risk-neutral domain.[3] In other words, the two puzzles reflect the willingness of investors to pay high risk premia for securities (index put options and senior CDO tranches) which provide insurance against a systematic crash of firm values. If there is an economic crisis, then the collapse of firm values will tend to affect all firms more or less equally, and stocks, bonds and related instruments will fall in value together.

[2] Note that we have drawn the base-correlation-skew graph in a reverse way, by placing the further out-of-the-money senior tranches to the left of the x-axis, in order to make it consistent with the index put options. To be consistent with the maturity of the CDO, the equity-volatility sneer has been extrapolated to a five-year maturity on this day, using the method explained later in the paper.

[3] The ratio of the distributions is the pricing kernel. The ratio rises as we move from higher-wealth to lower-wealth outcomes. See, for example, Rosenberg and Engle (2002) and Bliss and Panigirtzoglou (2004).

In order to explain the base-correlation skew, many researchers have tried to retain the parsimony of the copula methodology but change the assumption that the underlying factor is Gaussian in distribution. Among the proposed extensions are the Gaussian copula model with cross-sector correlations (Gregory and Laurent 2005), the Gaussian copula model with conditional factor loadings (Andersen and Sidenius 2005), and non-Gaussian copulas such as the Student-t copula (Hull and White 2004) and Archimedean copula (Rogge and Schoenbucher 2003). Hull and White (2006) dispense with base correlations altogether and imply from the data that particular copula distribution which fits the market data perfectly. Models with these extensions are statistically appealing in terms of capturing certain aspects of empirical observations, but they have limited economic justification and are not universally accepted. One paper which we have discovered after completing our work, by Coval, Jurek and Stafford (2009a), uses a similar insight to ours, namely that equity-index options can be used to help price tranches of the CDX index. Another paper by Collin-Dufresne, Goldstein and Yang (2012) also takes the option-market approach. Our approach is simpler than theirs, as we retain the copula methodology which they do not.

Our main hypothesis is that the "market systematic factor" used (but not explicitly defined) in the one-factor copula framework is the same as the market systematic factor used in modelling equity returns. To test this, we develop an "equity-implied copula model" in which we replace the Gaussian copula with one whose distribution is implied by equity-index puts. We then demonstrate that our equity-implied copula model can generate plausible spreads for different CDO tranches, without the need for a pronounced base-correlation skew. We also show that each week from October 2006 to November 2007 the implied equity-return distributions, on which the model is based, have very similar moments to the fat-tailed credit-return distributions implied from spreads on CDO tranches, confirming that the CDO market takes account of the same factors as the equity market. Nevertheless, despite the similarity of these two implied distributions, our equity-implied copula model generates spreads on senior CDO tranches before the subprime crisis which are three to four times larger than those observed in the market. After the crisis begins, model and market spreads become much closer in size. It appears that before the crisis the kernel used to price

CDOs is not consistent with the kernel used to price equity-index options, but afterwards it is. [4]

Our paper is organized as follows. Descriptions of the data on CDO tranches and on equity-index options are given in Section 2. Section 3 presents the equity-implied copula model and the methodology used for implying the risk-neutral credit-loss distribution and the equity-return distribution. Both theoretical and empirical results are presented in Section 4. Section 5 draws together the conclusions and implications of this study.

2. Data Description

To conduct this study we require data on CDO tranches and equity-index put options with the same maturity. For the CDO tranches, we use the 5-year standardised CDO tranches written on the CDX NA IG (North American Investment Grade) Index with 125 most-liquid names. For the put options, we use the short-term S&P 500 options traded on the CBOE and extrapolate them to a 5-year maturity to match the CDOs. In the two sub-sections which follow, we describe the CDO tranche data and the index-option data.

2.1 Standard CDO Tranches

CDO tranches began active trading in the 1990s, but reliable data on US index prices only became available after Dow Jones introduced the five-year CDX NA IG index (CDX, for short) in October 2003. Our data come from Datastream and comprise weekly closing values for the CDX index and its tranches from October 2006 to November 2007 (59 usable weekly observations). This period includes both the peak of the credit bubble and the onset of the subprime mortgage crisis in July 2007. The underlying basket of 125 firms in the index is revised every March and September, so our index data comprise three individual series, denoted by CDX i, $i = 7, 8 \; and \; 9$.

[4] Coval et al (2009b) reach a similar conclusion. This has been contested by Collin-Dufresne, Goldstein and Yang (2012) in a very recent paper, who argue that early default can have a substantial effect on the results and this needs to incorporated in the model.

For the descriptive purposes of this section, we knit together these three series. Longstaff and Rajan (2008) use proprietary data on the same index for October 2003 to October 2005, which will allow us to make comparisons with that earlier period.

The CDX tranche data (i.e. the CDO data) are weekly closing quotes for the 0-3, 3-7, 7-10, 10-15, and 15-30 percent tranches. The prices for tranches, with the exception of the equity 0-3 tranche, are quoted in basis points of premium paid to a CDO investor for absorbing the losses associated with the corresponding tranche. Consider, for example, a CDO with a nominal value of $100 million. A price of 135 for the 3-7 junior mezzanine tranche implies that an investor will receive a premium of 135 basis points per year (paid quarterly) on the $4 million capital in this tranche and, in return, the investor agrees to absorb any losses on the underlying CDX index in excess of $3 million and up to $7 million. Similarly, an investor who buys the 7-10 mezzanine tranche is paid (for example) 35 basis points and absorbs CDX losses between $7 million and $10 million. The 0-3 equity tranche is generally quoted (by market convention) in terms of a percentage of the tranche paid immediately plus 500 basis points per annum. However, to facilitate comparisons with other tranches we will convert this into basis points. [5]

Table 1 gives summary statistics for the spread on the index and for spreads on the individual tranches. The average CDX index spread is 45 basis points, which is less than the 55 basis points reported by Longstaff and Rajan for the earlier 2003-2005 period, despite the fact that our sample includes four months of the subprime crisis. The average spread for the most junior, 0-3 equity tranche is 1341 basis points, which is converted from an average number of points up-front of 30.72%. This spread is approximately ten times as large as that on the next tranche, the 3-7 junior mezzanine tranche, which is 135 basis points. The average spreads diminish with seniority, being

[5] For example, a price of 22.94% for the equity tranche means that an investor would receive 22.94% upfront of the tranche's notional value, plus a premium of 500 basis points per year (paid quarterly, on the outstanding notional value), in order to absorb the first three percent of losses on the CDX index. In our example, the spread on this tranche is then 500 basis points plus (if an annual discount rate of 5% is applied) an additional 530 per annum (five payments of which sum to a present value of 2294), i.e. 1030 basis points per annum.

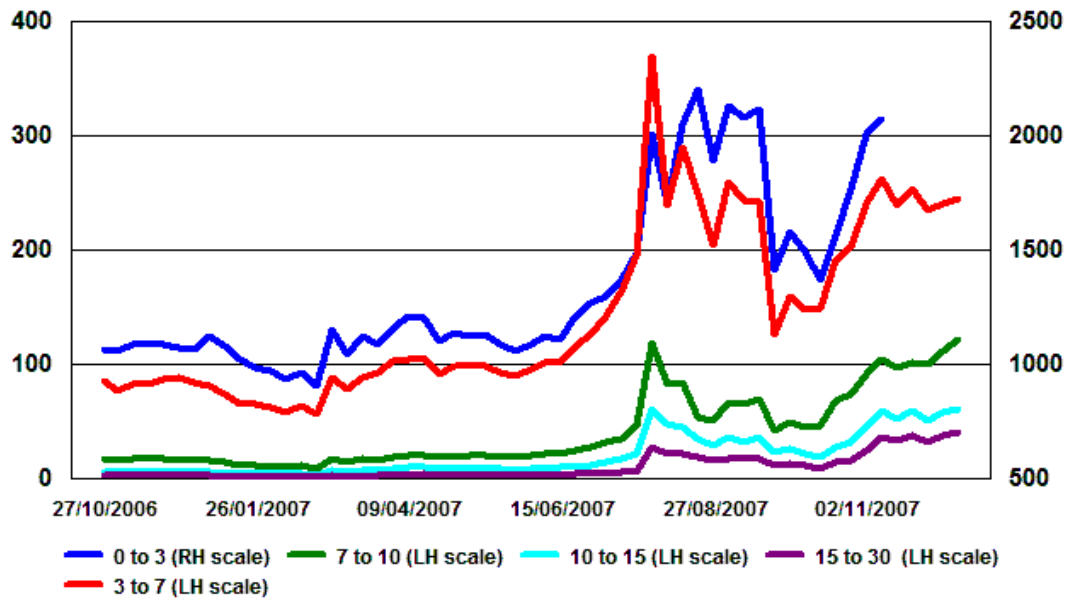
35 basis points for the 7-10 tranche, 17 basis points for the 10-15 tranche, and only 8 basis points for the 15-30 tranche.[6]

[Table 1 here]

Figure 2A plots the time series of spreads for different tranches. Note that the 0-3 equity tranche is plotted with the right-hand scale in the figure and the other tranches with the left-hand scale. All spreads are relatively stable until the onset of the crisis in July 2007, at which time they show large jumps. In particular, the 15-30 senior tranche has a spread of only a few basis points before that date and then jumps to about 25 basis points.

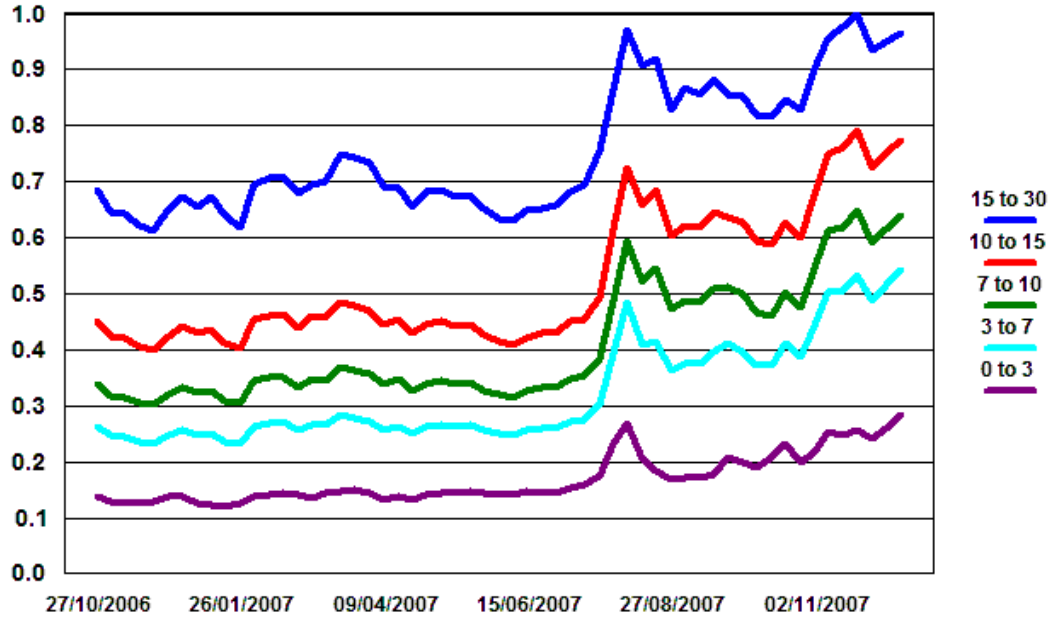
Figure 2B plots the base correlations required by the Gaussian copula to give the observed market spreads on each sample date. The base-correlation skew is very apparent, with the 0-3 equity tranche at the bottom of the figure priced at a correlation of only 15-25% and the 15-30 senior tranche at the top of the figure priced at a correlation of 60-100%. As the diagram shows, when the subprime crisis starts in July 2007, all of these correlations take an upward jump of 10 to 20 percentage points, although the jump is smaller for the 0-3 equity tranche than for the others.

Figure 2A Time Series of Spreads for 5-year Standardised CDX Tranches in basis points
(right-hand scale for 0-3 equity tranche, left-hand scale for other tranches)



[6] Longstaff and Rajan (2008) report for the earlier October 2003 to October 2005 period that spreads are 1759, 240, 82, 34 and 12 basis points for the 0-3, 3-7, 7-10, 10-15 and 15-30 tranches respectively. Comparing with September 2004 to September 2007 in Collin-Dufresne et al (2012), our sample spreads are almost exactly the same as theirs, except for the 0-3 tranche for which they report 1472 basis points.

Figure 2B Time-Series of Base Correlations for CDX Tranches required by the Gaussian Copula to give market spreads



Source: own calculations, based on the standard Gaussian copula model with market prices

2.2 Daily Short-Term Index Put Options on S&P 500

Data on S&P 500 put options from Oct 2006 to Nov 2007 are obtained from DataStream for all of the available exercise prices and maturities. We choose dates once per week which match the CDO data. The S&P 500 (SPX) index options are European style and can only be exercised on the last trading day. The underlying asset is the index, but for calculating option prices traders consider the futures contract on the S&P 500 (with the same maturity as the option) to be the relevant asset.

Figure 3A plots the time-series of the S&P 500 closing prices over this period. The market is generally rising, but also shows large falls in February and July 2007.

Figure 3A Historical Prices of S&P 500 Index

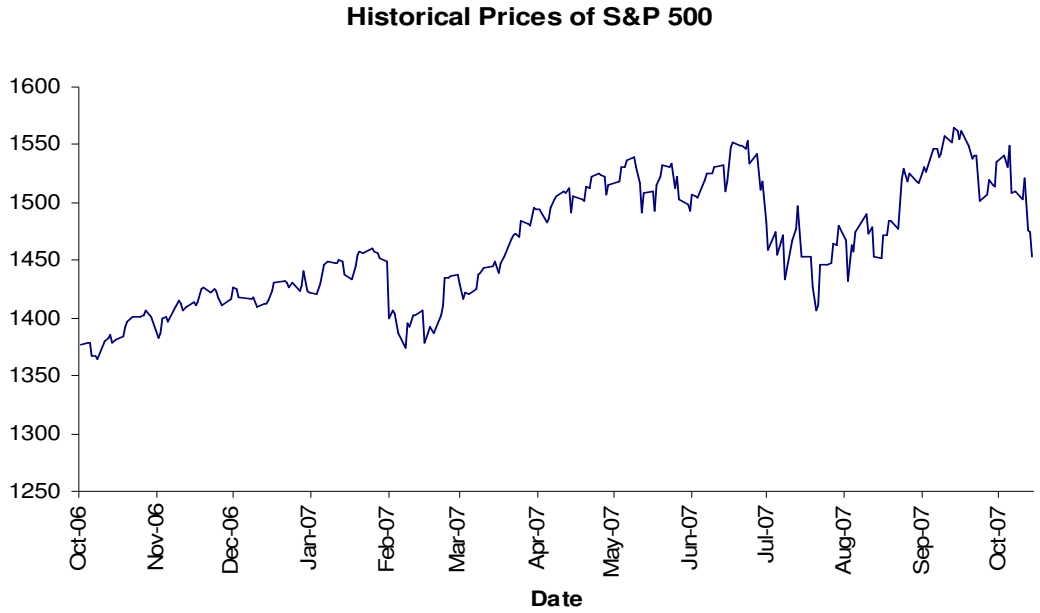
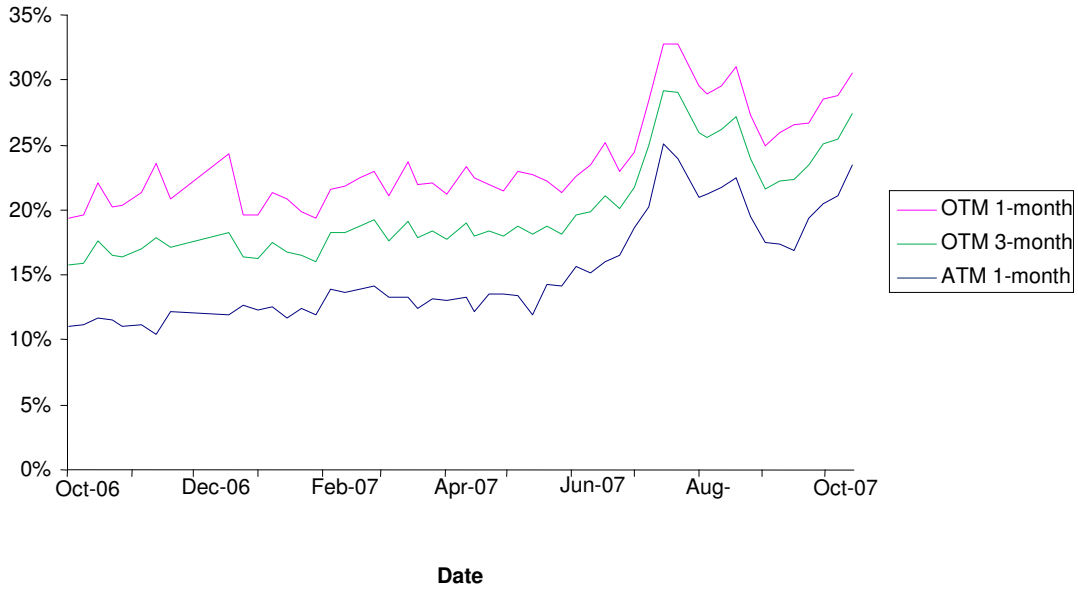


Figure 3B plots the implied volatilities for the index options over this period and Table 2 gives descriptive statistics of these data. The at-the-money (ATM) 1-month implied volatility (at the bottom of the figure) is initially in the 10-15% range, which is low on a historical basis. The out-of-the-money (OTM)[7] put-option volatility for 1-month (at the top of the figure) is considerably higher, being 20-25% initially, indicating that options with lower moneyness have higher implied volatilities (the volatility sneer). Comparing OTM 1-month options in the top plot with OTM 3-month options in the middle plot, the latter have lower volatilities and this demonstrates that implied volatilities decline with increasing maturity. The stock-market fall in February 2007 has little impact on implied volatilities, but the fall in July 2007 (when the subprime crisis starts) has quite a large impact, with the ATM volatility approximately doubling.

[7] OTM here is for a moneyness of 0.8 relative to the forward price.

Figure 3B Time Series of Implied Volatilities for S&P500 Index Options



[Table 2 here]

3. Model and Methodology

We proceed in three steps. First (in section 3.1), we present the methodology for implying the risk-neutral equity-return distribution from the option prices and outline our method for extrapolating long-term volatility surfaces from short-term surfaces. Second (in section 3.2), we show how to imply the risk-neutral credit-loss distribution from the CDO tranche prices. And third (in section 3.3), we develop the equity-implied copula model which is the main focus of this research.

3.1 Implying the Risk-Neutral Equity Return Distribution

In this subsection we present the methodology[8] for implying the risk-neutral equity return distribution from the European option prices and extrapolating to longer maturities. The underlying principles are that the probability density depends on the second derivative of the option price with respect to exercise price (Breeden and

[8] Our methodology is based on Jackwerth and Rubinstein (1996) and Breeden and Litzenberger (1978).

Litzenberger, 1978) and, to make that derivative continuous, option prices need to be estimated (from interpolated or extrapolated volatilities) for the relevant maturities.

Let S_t denote the underlying share (index) price at t and K denote the strike price of a European put option with maturity T. Define $B(t,T)$ as the price at time t of a zero-coupon bond that pays off \$1 at time T and let E_T denote the forward risk-neutral expectations with respect to $B(t,T)$ for time T.

Under the forward risk neutral measure with respect to $B(t,T)$, the value of the put option at time 0 can be written as:

$$p = B(0,T) \int_0^{S_T=K} (K - S_T) f(S_T) dS_T$$

Assume that prices of put options are continuous at different levels of strike prices K. Differentiate the option prices with respect to the strike price K, then we obtain

$$\frac{\partial p}{\partial K} = B(0,T) \int_0^{S_T=K} f(S_T) dS_T = B(0,T) \Pr ob(S_T \leq K) \qquad (1)$$

This gives us the risk-neutral distribution function of the underlying share with respect to $B(t,T)$. Then differentiate Equation 1 again with respect to the strike price K to give:

$$\frac{\partial^2 p}{\partial K^2} = B(0,T) f(K)$$

This shows that the risk-neutral density function f of the underlying share is given by

$$\Pr ob(S_T = K) = f(K) = \frac{1}{B(0,T)} \frac{\partial^2 p}{\partial K^2} \qquad (2)$$

Since the expectation of the underlying share price at time T under the risk-neutral measure with respect to $B(t,T)$ is its forward price F, the value of put option with strike K and time to maturity T is given by:

$$p_K = B(0,T)\big(K \cdot \Phi(-d_2) - F \cdot \Phi(-d_1)\big)$$

where $d_1 = \dfrac{\ln(F/K) + \frac{1}{2}\sigma_{K,T}^2 T}{\sigma_{K,T}\sqrt{T}}$, $d_2 = d_1 - \sigma_{K,T}\sqrt{T}$, $\sigma_{K,T}$ is the implied volatility from market price of option with strike K and time to maturity T, and Φ denotes the cumulative normal distribution.

Define p' as:

$$p_K' = \frac{p}{B(0,T)} = K \cdot \Phi(-d_2) - F \cdot \Phi(-d_1) \tag{3}$$

We can now rewrite the risk-neutral distribution of the underlying share at time T as:

$$\Pr ob(S_T = K) = f(K) = \frac{\partial^2 p'}{\partial K^2} \tag{4}$$

Equation 4 is important because it indicates that we can imply the risk-neutral distribution of S_T for a given maturity from a set of put option prices at different exercise prices, i.e. from the volatility smile at that maturity.

Assume that the implied volatility σ_K is a function of strike K at a given time to maturity T, and assume that σ_K is twice differentiable with respect to strike K. Then we can rewrite Equation 3 as:

$$p_K' = K\Phi(-g_k) - F\Phi\left(-(g_K + \sigma_K\sqrt{T})\right) \tag{5}$$

where $g_K = \dfrac{\ln(F/K) - 0.5\sigma_K^2 T}{\sigma_K\sqrt{T}}$ and is twice differentialable with respect to strike K.

Given this, the first-order derivative of the option price with respect to K becomes:

$$\begin{aligned}\frac{\partial p_K'}{\partial K} &= \Phi(-g_K) - K\varphi(-g_K)g_K' + F\varphi\left(-(g_K + \sigma_K\sqrt{T})\right)\cdot(g_K' + \sigma_K'\sqrt{T}) \\ &= 1 - \Phi(g_K) + K\varphi(-g_K)\sigma_K'\sqrt{T} \\ &= 1 - \Phi(g_K) + K\varphi(g_K)\sigma_K'\sqrt{T}\end{aligned}$$

where φ is the normal-distribution density function. And the second-order derivative is then:

$$\frac{\partial^2 p_K^{'}}{\partial K^2} = -\varphi(g_K)g_K^{'} + \varphi(g_K)\sigma_K^{'}\sqrt{T} + K\sqrt{T}\left(-g_K\varphi(g_K)g_K^{'}\sigma_K^{'} + \varphi(g_K)\sigma_K^{''}\right)$$
$$= \varphi(g_K)\left(\sigma_K^{'}\sqrt{T} - K\sqrt{T}g_K g_K^{'}\sigma_K^{'} + K\sqrt{T}\sigma_K^{''} - g_K^{'}\right) \quad (6)$$

where $g_K^{'} = \dfrac{(-\frac{1}{K} - T\sigma_K\sigma_K^{'})\sigma_K\sqrt{T} - (\ln(F/K) - 0.5\sigma_K^2 T)\sigma_K^{'}\sqrt{T}}{\sigma_K^2 T}$.

This derivation gives the risk-neutral density, as we know from (4) that $\Pr ob(S_T = K) = f(K) = \dfrac{\partial^2 p'}{\partial K^2}$.

In the Black-Scholes model, where volatility is assumed to be a constant across both strikes and maturities, i.e. $\sigma_K^{'} = 0$ and $\sigma_K^{''} = 0$, the equivalent to (6) is

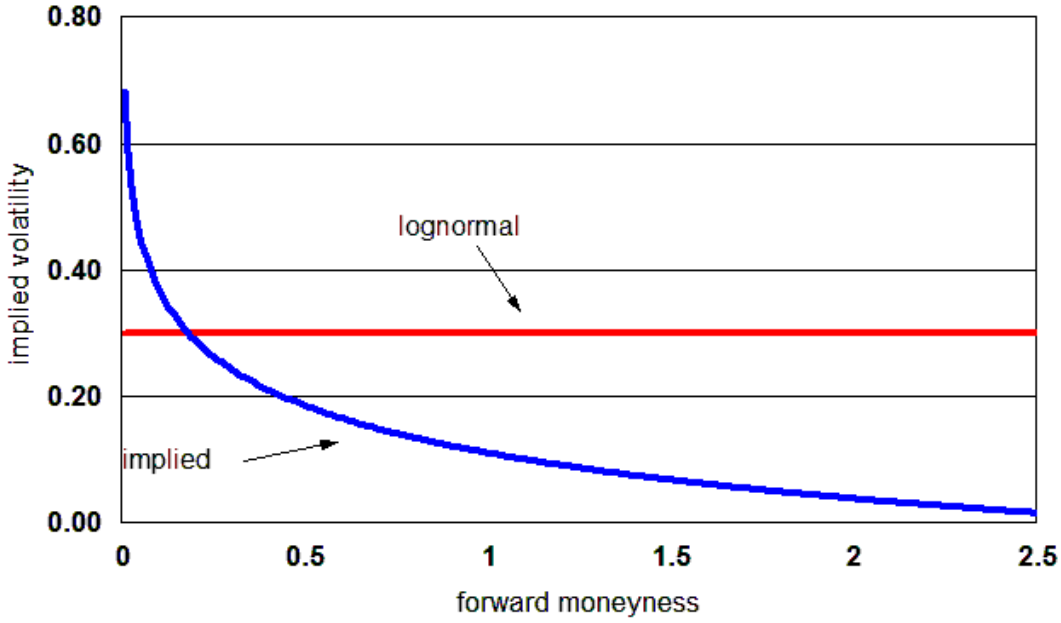
$$\frac{\partial^2 p_K^{'}}{\partial K^2} = \varphi(g_K)(-g_K^{'}) = \frac{1}{\sqrt{2\pi T}}\frac{e^{-\left(\ln(F/K) - \frac{1}{2}\sigma^2 T\right)^2/(2\sigma^2 T)}}{K\sigma},$$

which reflects the log-normal distribution assumption of the underlying share prices.

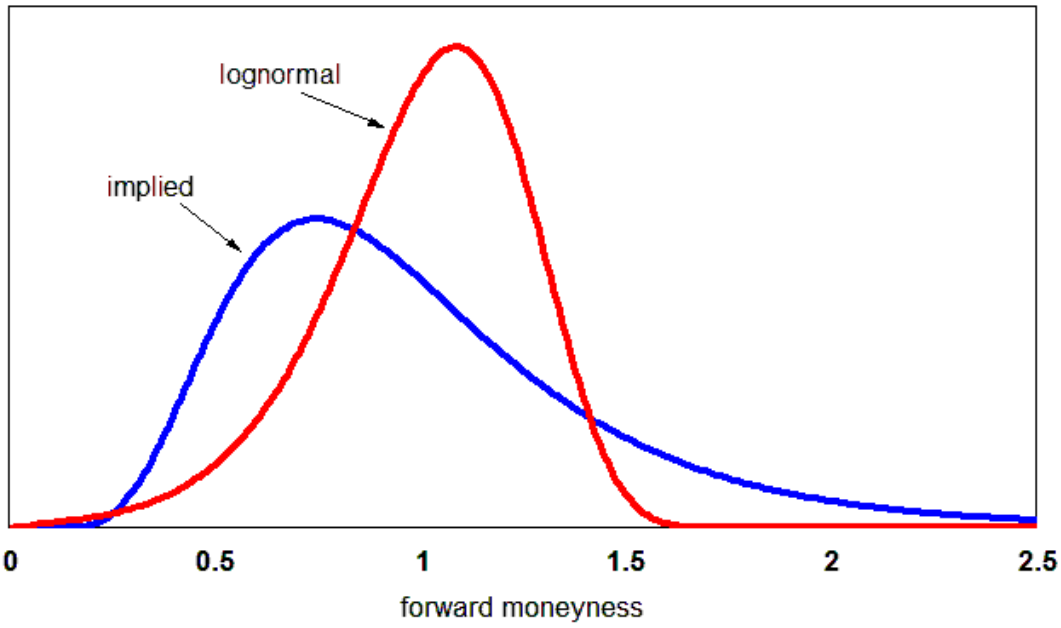
The upper plot in Figure 4 compares the Black-Scholes constant volatility with the volatility smile using SPX data on 27th October 2006. For consistency with CDOs we have extrapolated the market data to a 5-year horizon for this diagram, using the method to be explained below. The lower plot in Figure 4 gives the corresponding implied risk-neutral return distribution. The latter has a much fatter left tail than the Black-Scholes (log-normal) one, reflecting one or both of: (i) expectations by traders of severe downside losses; and/or (ii) a large risk premium being required on potential downside losses.

Figure 4 Volatility Smiles and Equity-Index Price Distributions on 27th October 2006, for T=5 years

Volatility Smiles



Equity-Index Distributions



We need to generate the risk-neutral density for the stock index using a wider set of exercise prices and a longer maturity (5 years) than that directly available from using

this procedure with market data. We therefore fit a volatility surface to the available data for each week and extrapolate. The function used is:

$$\sigma_{K,T} = \hat{\beta}_0 + \frac{\hat{\beta}_1}{\sqrt{T}} \ln\left(\frac{K}{F_T}\right) + \frac{\hat{\beta}_2}{T}\left(\ln\left(\frac{K}{F_T}\right)\right)^2$$

where $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\beta}_3$ are calibrated by least squares to the observed short-term options. There are typically five short-term maturities and fifteen exercise prices on any day. The resulting R-squared for the in-sample fit has a median of 98% and the worst day's regression has an R-squared of 89%, so the fit to the short-term data is good. Related work by Gemmill and Yang (2010) tests the performance of extrapolation, using both short-term and long-term sets of data on UK options; it confirms that extrapolation works well, but that coefficients need to be estimated on each day because they vary over time.

3.2 Implying the Risk-Neutral Credit Loss Distribution with the Gaussian Copula

In this section, we present the methodology for implying the risk-neutral credit-loss distribution from spreads on the first-loss CDO tranches (i.e. on tranches which absorb all losses from zero to the specified level). We beginby specifying the credit-loss density function and then we show how that density can be revealed with a Gaussian copula.

The first-loss tranche of a CDO is one which starts at 0 and bears losses up to a particular detachment level. For example, we may define not only the 0-3 first-loss tranche, but also join together the 0-3 and 3-7 tranches to give a 0-7 first-loss tranche. In general, any first-loss tranche of a CDO with a level of detachment l^+ can be replicated by another first-loss tranche with level of detachment l^- together with a (l^-, l^+) mezzanine tranche. Let l_t denote the portfolio loss at time t and let h_l be the premium (spread) of the $(0, l)$ first-loss tranche.

The present value of payments on the premium leg is:

$$PV(premium) = h_l \cdot \int_0^T B(0,t) \frac{l - l_t}{l} 1_{\{l_t \le l\}} dt$$

The present value of payments on the contingent leg is:

$$PV(contigent) = \int_0^T B(0,t)d\left(1 - \frac{l - l_t}{l} 1_{\{l_t \le l\}}\right)$$
$$= -\int_0^T B(0,t)d\left(\left(\frac{l - l_t}{l}\right) 1_{\{l_t \le l\}}\right)$$

Equating the two payments gives

$$h_l \cdot \int_0^T B(0,t)\left(\frac{l - l_t}{l}\right) 1_{\{l_t \le l\}} dt = -\int_0^T B(0,t)d\left(\left(\frac{l - l_t}{l}\right) 1_{\{l_t \le l\}}\right)$$

Differentiating the above equation with respect to t, we have

$$h_l \cdot \left(\frac{l - l_t}{l}\right) 1_{\{l_t \le l\}} dt = -d\left(\left(\frac{l - l_t}{l}\right) 1_{\{l_t \le l\}}\right)$$

The solution of the above ordinary differential equation is

$$(l - l_t) 1_{\{l_t \le l\}} = l e^{-h_l t} \tag{7}$$

By differentiating Equation 7 with respect to l, the implied credit-loss distribution function for a given maturity T is given by

$$\Pr ob(l_T \le l) = e^{-h_l T} - l T e^{-h_l T} \frac{\partial h}{\partial l} \tag{8}$$

The implied credit-loss density function for a given maturity T is then given by

$$\Pr ob(l_T = l) = T e^{-h_l T}\left(l T\left(\frac{\partial h}{\partial l}\right)^2 - l \frac{\partial^2 h}{\partial l^2} - 2\frac{\partial h}{\partial l}\right) \tag{9}$$

Similar to the market quote of an equity option, which can be calculated using the Black-Scholes formula given the implied volatility, the market quote of a first-loss CDO tranche can be calculated using the one-factor Gaussian copula model given the implied base correlation.

Let $\overline{p}$ denote the probability of default implied from the CDX index spread and ρ_l be the base correlation implied from the $(0,l)$ first-loss CDO tranche on the CDX index. In the one-factor Gaussian copula model, the conditional expected portfolio loss with a constant recovery rate δ is

$$l(y) = (1-\delta)\Phi\left(\frac{\Phi^{-1}(\overline{p}) - \sqrt{\rho_l}\, y}{\sqrt{1-\rho_l}}\right)$$

The conditional expected tranche survival then becomes

$$TS(y) = \left(\frac{l - l(y)}{l}\right) 1_{\{l(y) \le l\}} = \left(1 - \frac{1-\delta}{l}\Phi\left(\frac{\Phi^{-1}(\overline{p}) - \sqrt{\rho_l}\, y}{\sqrt{1-\rho_l}}\right)\right)_+$$

and the unconditional expected tranche survival is given by

$$TS = \int_Y \left(1 - \frac{1-\delta}{l}\Phi\left(\frac{\Phi^{-1}(\overline{p}) - \sqrt{\rho_l}\, y}{\sqrt{1-\rho_l}}\right)\right)_+ \varphi(y)dy$$

Given Equation 7, the premium on the T-year maturity $(0,l)$ first-loss CDO tranche using the Gaussian copula is

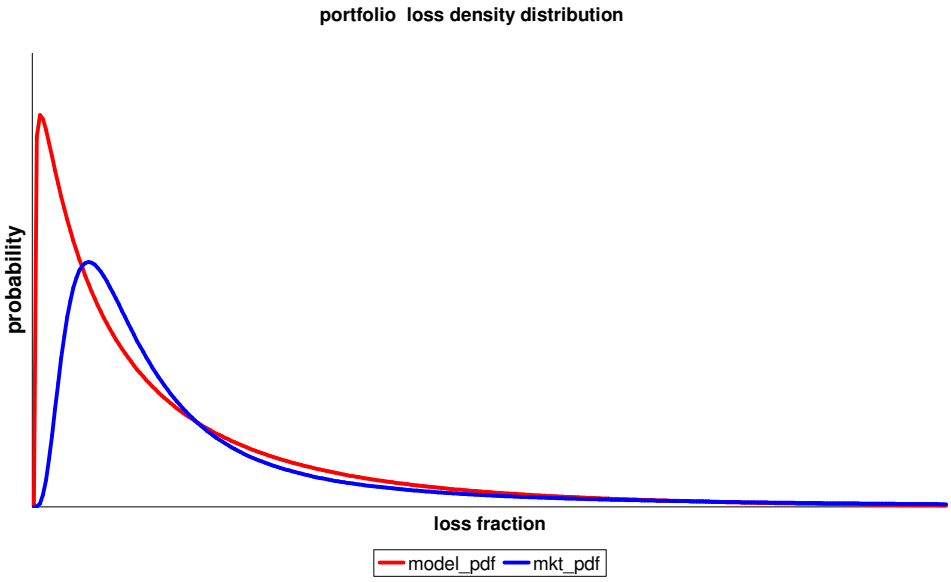
$$h_l = -\frac{1}{T}\ln\left(\int_Y \left(1 - \frac{1-\delta}{l}\Phi\left(\frac{\Phi^{-1}(\overline{p}) - \sqrt{\rho_l}\, y}{\sqrt{1-\rho_l}}\right)\right)_+ \varphi(y)dy\right) \tag{10}$$

We can then imply the risk-neutral credit-loss distribution using the credit-loss density function (9) and the premium for a Gaussian copula (10).

Figure 5 compares the Gaussian-copula risk-neutral credit-loss distribution implied from CDX tranches on 27th October 2006 with the Gaussian-copula distribution that would arise if the base correlation were a constant 25% across tranches.[9] The market implied distribution (blue line) is more right-skewed and exhibits a longer tail for severe losses than the simple one with no correlation skew (the red line). In other words, the market requires a larger spread on senior tranches than can be generated by a Gaussian copula with constant correlation.

[9] Note that we assume linearity between implied base correlations at different detachment points in order to obtain the whole implied distribution.

Figure 5 Market-Implied Credit Loss Distribution vs. Simple Gaussian-Model Implied Credit Loss Distribution on 27th October 2006



3.3 The Equity-Implied Copula Model

In this section, we explain how to implement a new copula model, which we call the equity-implied copula, that uses the distribution of the systematic factor implied by the prices of equity-index options.

Let θ_t denote the level of underlying index and T denote the selected time horizon. The expected value of $\ln(\theta_T)$ under the measure with respect to $B(t,T)$ is then given by:

$$E_T\left(\ln(\theta_T)\right) = \int_0^\infty \ln(K) f(K) dK$$

Similarly, the variance of $\ln(\theta_T)$, given the probability density function, is

$$\begin{aligned} Var_T\left(\ln(\theta_T)\right) &= E_T\left[\ln(\theta_T)^2\right] - \left(E_T\left[\ln(\theta_T)\right]\right)^2 \\ &= \int_0^\infty \left(\ln(K)\right)^2 f(K) dK - \left(\int_0^\infty \ln(K) f(K) dK\right)^2 \end{aligned}$$

where $f(K)$ is given by Equation 6.

Now let us define the systematic factor Y, which is usually interpreted as the state of the economy, as

$$Y = \frac{\ln(\theta_T) - E_T[\ln(\theta_T)]}{\sqrt{Var_T[\ln(\theta_T)]}} \tag{11}$$

Y has a standardized distribution, with zero mean and unit variance. Thus the probability density function of Y under the risk-neutral measure with respect to $B(t,T)$ becomes:

$$f_Y(y) = be^{a+by} f(e^{a+by}) \tag{12}$$

where $a = E_T(\ln(\theta_T))$ and $b = \sqrt{Var_T(\ln(\theta_T))}$.

We know that the implied systematic factor Y from the index options exhibits a long left-hand tail relative to a Gaussian one, reflecting the skewness of the risk-neutral equity-index distribution; an example has already been given in Figure 4. To keep the parsimony of the one-factor copula methodology, we assume that firms within the underlying portfolio are homogeneous so that there is just one correlation, i.e. $\rho_i = \rho_j = \rho$ for all firms i and j. We can then define the default indicator X, which is referred-to in our paper as the asset value of the firm, as

$$X = \sqrt{\rho}\,Y + \sqrt{1-\rho}\,\varepsilon \tag{13}$$

where ε is the idiosyncratic factor and is independent of the systematic factor Y. ε is standard normally distributed with zero mean and unit variance while the distribution of Y is given by Equation 12.

As discussed earlier, losses on the tranches are contingent on the total losses on the underlying portfolio. Thus the distribution of total portfolio losses is sufficient for valuing tranches. Accordingly, our model can be regarded as an approach of modelling the distribution of total portfolio losses directly. Note that we are not implying that individual firm-level default information is unimportant. Rather, we suggest that for credit derivatives that are tied to a large portfolio, the approach of modelling portfolio-level losses directly may be simpler, more transparent and more tractable, with little loss in our ability to capture the underlying economics. In general,

the more diversified is the portfolio, the more applicable will be the aggregate loss approach taken here.

Letting $\overline{Y} = \sqrt{\rho}Y$ and $Z = \sqrt{1-\rho}\varepsilon$, Equation 13 can be rewritten as

$$X = \overline{Y} + Z$$

By the total probability theorem, we have

$$f_X(x) = \iint_{\overline{y}z} f_{X,\overline{Y},Z}(x, \overline{y}, z)d\overline{y}dz$$

Given the fact that Y and ε are independent, we have

$$f_X(x) = \iint_{\overline{y}z} f_{\overline{Y}}(\overline{y}) f_Z(z) f_X(x \mid \overline{y}, z)d\overline{y}dz$$

$$f_X(x) = \iint_{\overline{y}z} f_{\overline{Y}}(\overline{y}) f_Z(z)\delta_X(x-(\overline{y}+z))d\overline{y}dz$$

By substitution, we have

$$f_X(x) = \int_{\overline{y}} f_{\overline{Y}}(\overline{y}) f_Z(x-\overline{y})d\overline{y}$$

The risk-neutral probability density function of $\overline{Y}$ can be implied from Equation 12

$$f_{\overline{Y}}(\overline{Y} = \overline{y}) = \frac{1}{\sqrt{\rho}} f_Y(Y = \frac{\overline{y}}{\sqrt{\rho}}) = \frac{b}{\sqrt{\rho}} e^{a+\frac{\overline{y}}{\sqrt{\rho}}b} f(e^{a+\frac{\overline{y}}{\sqrt{\rho}}b})$$

Since Z is normally distributed with zero mean and $1-\rho$ variance, we have

$$f_Z(Z = z) = \frac{1}{\sqrt{1-\rho}}\varphi(\frac{z}{\sqrt{1-\rho}}) = \frac{1}{\sqrt{2\pi(1-\rho)}} e^{-\frac{z^2}{2(1-\rho)}}$$

Thus, the risk-neutral probability density function of the default indicator X is given by:

$$\begin{aligned} f_X(X = x) &= \frac{b}{\sqrt{2\pi\rho(1-\rho)}} \int_{-\infty}^{\infty} e^{a+\frac{\overline{y}}{\sqrt{\rho}}b} f(e^{a+\frac{\overline{y}}{\sqrt{\rho}}b}) e^{-\frac{(x-\overline{y})^2}{2(1-\rho)}} d\overline{y} \\ &= \frac{b}{\sqrt{2\pi(1-\rho)}} \int_{-\infty}^{\infty} e^{a+by} f(e^{a+by}) e^{-\frac{(x-\sqrt{\rho}y)^2}{2(1-\rho)}} dy \end{aligned} \tag{14}$$

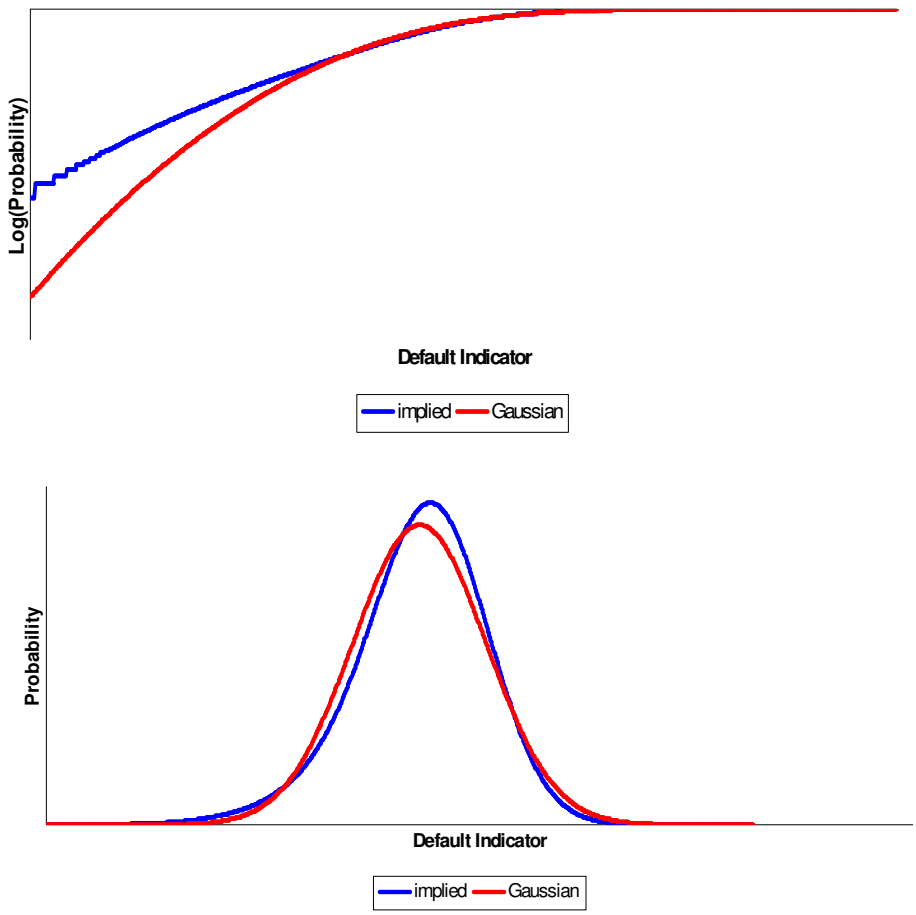
Similarly, the risk-neutral probability distribution function of the default indicator X is given by:

$$F_X(X \le x) = \frac{b}{\sqrt{2\pi(1-\rho)}} \int_{-\infty}^{x}\int_{-\infty}^{\infty} e^{a+by} f(e^{a+by}) e^{-\frac{(x-\sqrt{\rho}y)^2}{2(1-\rho)}} dydx \qquad (15)$$

where $a = E_T(\ln(\theta_T))$ and $b = \sqrt{Var_T(\ln(\theta_T))}$.

As an example, we continue to use data from 27th October 2006. We take the option-implied equity-return distribution from those data (shown earlier in Figure 4) and assume a correlation coefficient of $\rho = 25\%$. The upper panel of Figure 6 plots the log probability of the distribution function for the default indicator X (blue line) and compares it with the one assumed in the one-factor Gaussian copula model under the same correlation parameter ρ (red line). The lower panel of Figure 6 shows these as density functions. The newly-defined default indicator X exhibits a longer left-hand tail than the one in the Gaussian copula model, reflecting market perceptions of a more severe downside risk than is generated by the Gaussian model.

Figure 6 Implied Probability Distribution Function (upper panel) and Density Function (lower panel) of Default Indicator X (at a correlation parameter of 25%, on 27th October 2006)



Next we derive the credit-loss distribution for the equity-implied copula. Let C denote the default threshold, i.e. $1_{X \le C} = 1_{\tau \le T} = \overline{p}$, where $\overline{p}$ is implied from the CDX index spread. Given Equation 15, we have

$$C = F_X^{-1}(\overline{p})$$

Similar to the one-factor Gaussian copula model, the conditional default probability given the realization Y = y becomes

$$p(y) = \Pr ob(X \le C \mid Y = y) = \Phi\left(\frac{F_X^{-1}(\overline{p}) - \sqrt{\rho}\, y}{\sqrt{1-\rho}}\right)$$

Under the large homogeneous portfolio assumption, we have

$$\Pr ob(\left|L - p(y)\right| > \varepsilon \mid Y = y) \to 0$$

Thus,

$$\Pr ob(L \le l) = E(\Pr ob(L \le l \mid Y)) = \int 1_{\{p(y) \le l\}} f_Y(y) dy$$

where L denotes the portfolio loss.

We define $y*$ as

$$y* = \frac{F_X^{-1}(\overline{p}) - \sqrt{1-\rho}\,\Phi^{-1}(l)}{\sqrt{\rho}} \tag{16}$$

Then the probability distribution of the portfolio loss becomes

$$\Pr ob(L \le l) = 1 - F_Y(y*) \tag{17}$$

And the probability density function of the portfolio loss is given by

$$\Pr ob(L = l) = \sqrt{\frac{2\pi(1-\rho)}{\rho}} f_Y(y*) e^{\frac{N^{-1}(l)^2}{2}} \tag{18}$$

Now we can revisit the one-factor Gaussian copula. As discussed earlier, the one-factor Gaussian copula model is in principle equivalent to an adaptation of Merton's

model, in which the underlying asset process θ_t is assumed log-normal, with constant interest rate and volatility. Thus, we have in the Gaussian copula

$$a = \ln(F) - \frac{1}{2}\sigma^2 T \text{ and } b = \sigma\sqrt{T}$$

The risk-neutral probability density function becomes

$$\Pr ob(\theta_T = K) = f(K) = \frac{1}{\sqrt{2\pi T}} \frac{e^{-(\ln K - \ln F + \frac{1}{2}\sigma^2 T)^2/(2\sigma^2 T)}}{K\sigma}$$

Similarly, the systematic factor Y can be defined as

$$Y = \frac{\ln(\theta_T) - \ln(F) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}},$$

which is standard normal distributed with zero mean and unit variance.

By plugging the above equations into Equation 12, we have

$$f_Y(y) = be^{a+by} f(e^{a+by}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

Similarly, Equation 14 becomes

$$f_X(x) = \frac{1}{2\pi\sqrt{\rho(1-\rho)}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\rho}} e^{-\frac{(x-y)^2}{2(1-\rho)}} dy = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

indicating the default indicator has a standard normal distribution.

The risk-neutral credit loss distribution is given by

$$\Pr ob(L \le l) = \Phi\left(\frac{1}{\sqrt{\rho}}(\sqrt{1-\rho}\Phi^{-1}(l) - \Phi^{-1}(\overline{p})\right)$$

and its probability density function is given by

$$\Pr ob(L = l) = \sqrt{\frac{(1-\rho)}{\rho}} \exp\left(\frac{\Phi^{-1}(l)^2}{2} - \frac{((\Phi^{-1}(\overline{p}) - \sqrt{1-\rho}\Phi^{-1}(l))^2}{2\rho}\right) \quad (19)$$

We can see that the density function of the Gaussian copula (19) is a special case of the density function of our equity-implied copula (18): the former assumes that the systematic factor (Y) is normally distributed whereas the latter uses the empirical distribution derived from the equity-index options.

Figure 7 compares the credit-loss probability distribution generated by our equity-implied copula model with the one generated by the Gaussian model on October 27th 2006, assuming a base correlation of 0.25. The loss distribution of the equity-implied copula is more 'kinked' than that of the Gaussian copula, with a larger probability of generating small losses, a lower probability of generating intermediate-size losses, and a larger probability of generating large losses.

Figure 7 The Credit Loss Distribution Generated by Equity-implied Copula Model vs. the One Generated by Gaussian Copula Model (at a correlation parameter of 25%, on October 27th 2006)

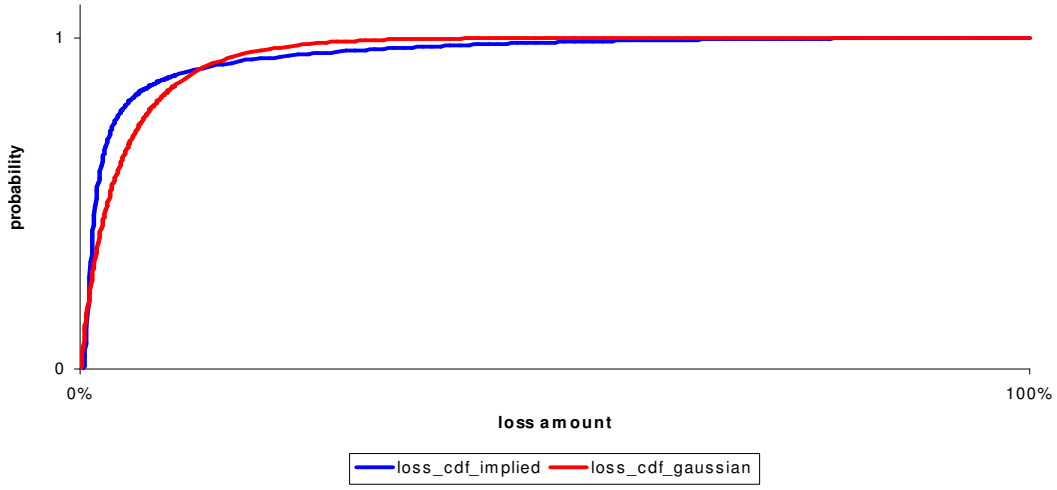
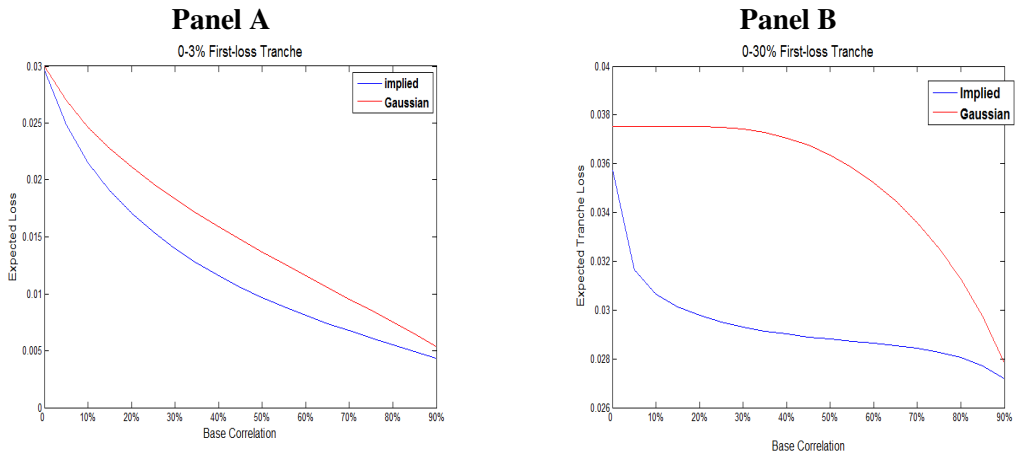


Figure 8 contrasts how expected losses are affected by the base correlation in the Gaussian copula model and in the new equity-implied copula model, using the data from October 27th 2006. The left-hand panel gives the plots for the 0-3 tranche and the right-hand panel gives the plots for the whole 0-30 tranche.

Figure 8 Sensitivity to Base Correlation of Equity-Implied Copula and Gaussian Copula for 0-3 and 0-30 Tranches (using data from October 27th, 2006)



For the 0-3 tranche in Panel A, to give expected losses of a particular size a slightly lower base correlation is required by our equity-implied copula model than by the Gaussian copula model. For example, for an expected loss of 1.5%, the base correlation of the 0-3% tranche in Panel A is 30% for the equity-implied copula and 45% for the Gaussian copula. Turning to the computed (but not traded) whole 0-30 tranche in Panel B, a much lower base correlation is required by the equity-implied copula for a given expected loss than by the Gaussian copula, except at extreme correlations. For example, for an expected loss of 3% the base correlation of the equity-implied copula is only 16% whereas the base correlation of the Gaussian copula is 85%.

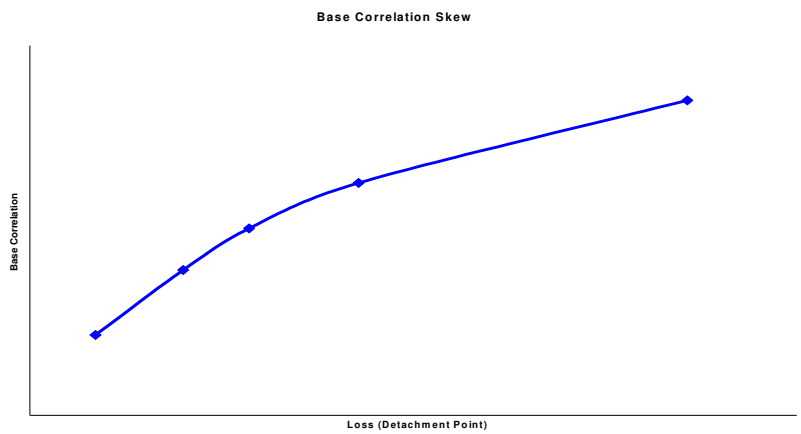
Let us suppose that the CDX index has an average default probability $\overline{p}$ of 5%, and a constant correlation of 25%. If we then price standard CDO tranches (0-3, 3-7, 7-10, 10-15 and 15-30) on October 27th 2006 using both the one-factor Gaussian copula and our equity-implied copula, the resulting premia are those given in Table 3. For the equity (0-3) tranche, the equity-implied copula generates a much smaller premium (1012 basis points) than the Gaussian copula (1658 basis points). For the mezzanine (7-10) tranche, the premiums are quite similar. For tranches above mezzanine, our equity-implied copula generates much higher spreads than the Gaussian copula. In particular, the equity-implied copula generates a spread of 42 basis points for the senior (15-30) tranche, which is more than four times larger than that of the Gaussian copula (which only generates 9 basis points).

[Table 3 here]

Using the spreads calculated by our equity-implied copula at a constant correlation of 25% given in the second column of Table 3, we now derive the base correlations which would be required for different tranches if the Gaussian copula had been used instead. These are given in Table 4 and also plotted in Figure 9. The implied base correlation required by the Gaussian copula increases monotonically as the detachment level increases, e.g. a base correlation of 45% is required to match the 0-3 tranche, whereas a base correlation of 61% is required to match the whole 0-30 tranche. This indicates that a constant correlation structure in our equity-implied copula can generate spreads for which the Gaussian copula requires a substantial base-correlation skew.

[Table 4 here]

Figure 9 Base Correlations Required by the Gaussian Copula to Match Prices from the Equity-Implied Copula (assuming an average default probability of 5% and a base correlation of 25% for the latter model)



4 Empirical Results

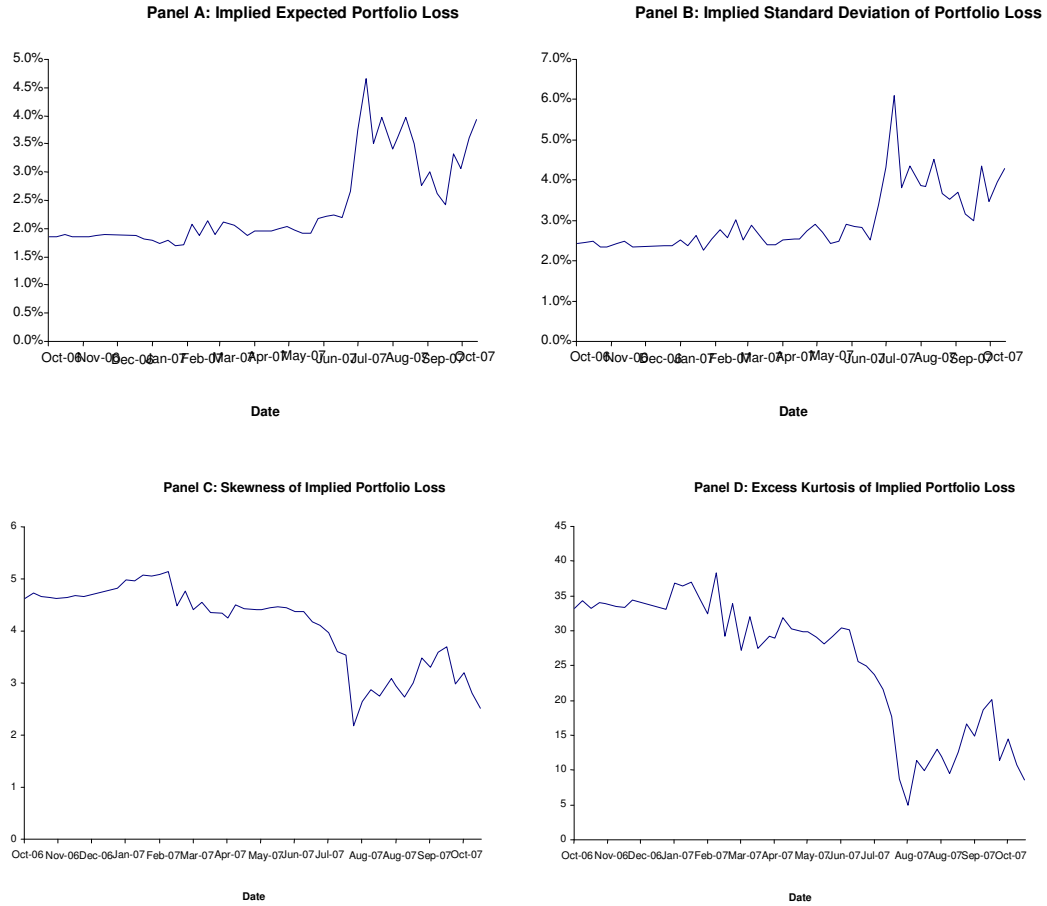
4.1 Credit Loss Distribution from the Gaussian Copula

Before using equity data and the new model, it may be helpful to examine how the shape of the credit-loss distribution changes over October 2006 to November 2007, using the conventional Gaussian copula and an asset correlation of 0.25. Figure 10, panel A, indicates that the expected portfolio loss on the CDX index is relatively

stable at around 2% before the July 2007 subprime crisis and then jumps to around 3.5% after the crisis begins. There are wide variations in expected losses after July 2007, resulting in a jump in the standard deviation of the implied risk-neutral loss distribution from about 2.5% before the crisis to a range of 3-6% (see Panel B).

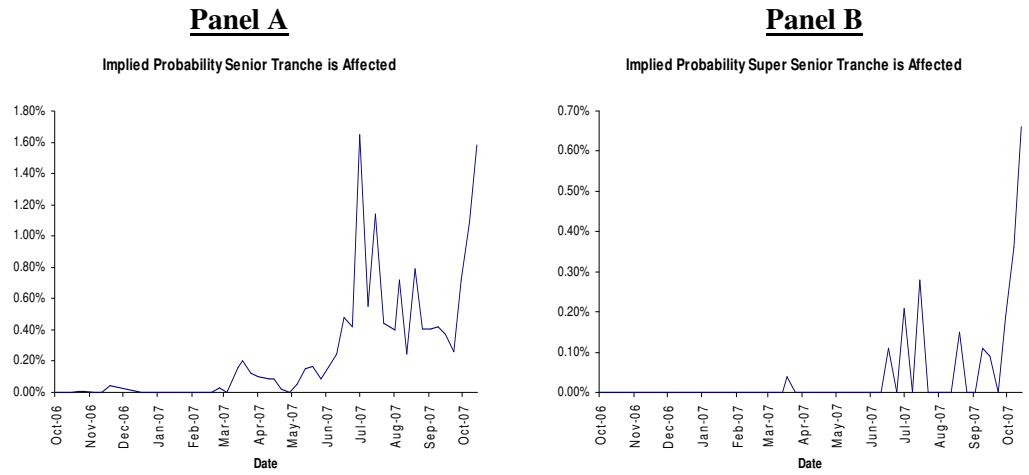
Before the crisis the measured skewness of the implied distribution is about 5 (panel C), indicating that the mass of the distribution is concentrated on the left of the distribution with a few relatively high values in the right-hand tail. After the crisis, the implied distribution is less skewed than before (skewness falls to about 3), with more outcomes concentrated on the right of the distribution, reflecting more scenarios of bigger losses expected by the market. Similarly, the excess kurtosis of the implied credit loss distribution is about 30 before the crisis (panel D), indicating that extreme losses are perceived to be infrequent, whilst the relatively lower level of kurtosis (approximately 10) after the crisis starts (in July 2007) reflects the fact that more and more extreme losses have now been priced into the tranches: what had been extraordinary before the crisis has now become more commonplace.

Figure 10 Moments of the Risk-Neutral Loss Distribution on the CDX Index each week, October 2006 to November 2007, (implied by fitting the Gaussian copula to the cross-section of tranche premia at a correlation of 0.25)



Continuing to use the Gaussian copula (because that is what traders used), it is interesting to compare the probabilities that losses will occur on the "safe" senior 15-30 with those on the "extremely safe" super-senior 30-100 tranches of the CDX index. Figure 11 plots the time-series of breach probabilities for these two tranches. Before the crisis (panel A), traders regard the senior 15-30 CDO tranche as being virtually riskless, but its breach probability jumps to more than 1.5% after the crisis starts. The super-senior 30-100 CDO tranche (see panel B) is perceived to be risk-free before the crisis, but thereafter the picture is mixed: in some weeks it remains risk-free, but in others it has a breach-probability of 0.5% or more.

Figure 11 Implied Breach Probabilities of Senior and Super-Senior CDX Tranches, October 2006 to November 2007, based on Gaussian Copula

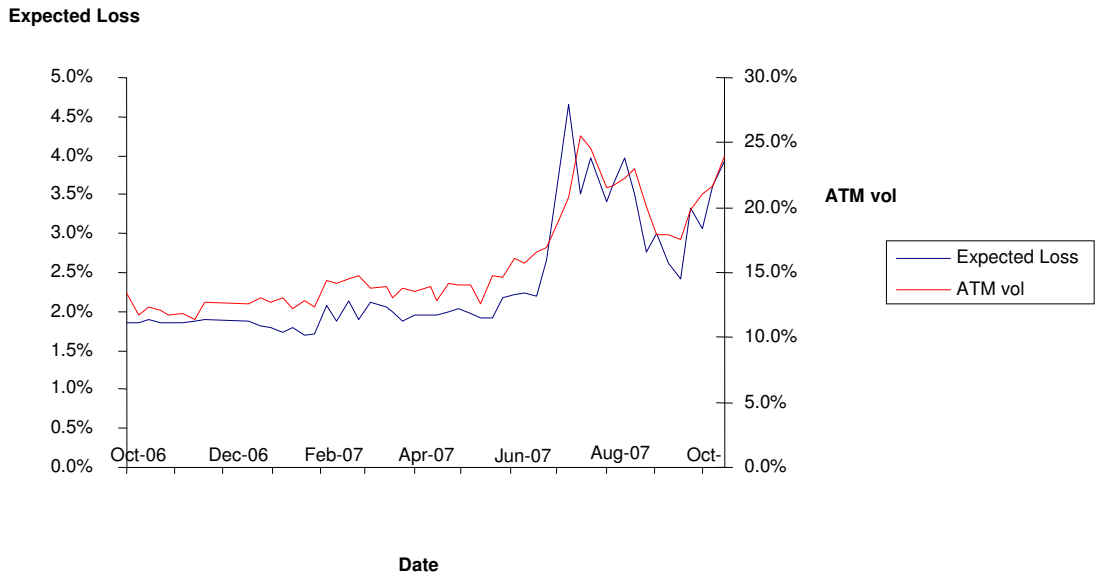


The most interesting question is whether the risk of losses on tranches above the most junior 0-3 equity tranche was underestimated by the market before the crisis. Coval et al (2009b) claim that it was, whereas Collin-Dufresne et al (2012) say that it was correctly priced. To answer this question, we make use of the equity-implied copula in the next sections of the paper.

4.2 CDOs and the Equity-Index Options

Before computing expected losses with the equity-implied copula, we will first confirm empirically that the CDX loss distribution is related to the risk-neutral distribution from equity-index options. Figure 12 plots the expected losses on the CDX index (already plotted in Figure 10), together with the at-the-money implied volatility from the index options (extrapolated to a maturity of five years). There is a remarkable consistency over time between the two series, with a correlation of 0.92. So the at-the-money index volatility does appear to contain information relevant to changes in the price of the CDX through time. That is not surprising, as the CDX represents spreads on a portfolio of bonds and we know from previous research that individual bond spreads and individual equity volatilities are closely related (see, for example, Campbell and Taksler, 2003). We also know from previous research that Merton's model for bond spreads, which is estimated using volatilities from the equity market, is useful for hedging changes in bond spreads, even if it underestimates the level of such spreads (Schaefer and Strebulaev, 2008).

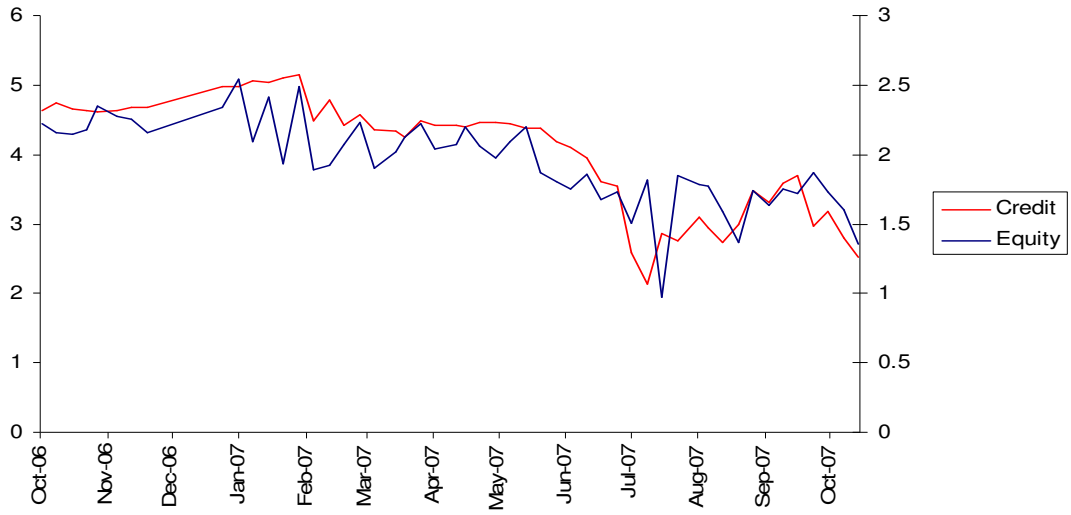
Figure 12 **Five Year Expected Credit Portfolio Loss on CDX vs. Five Year ATM Implied Equity Index Volatility**



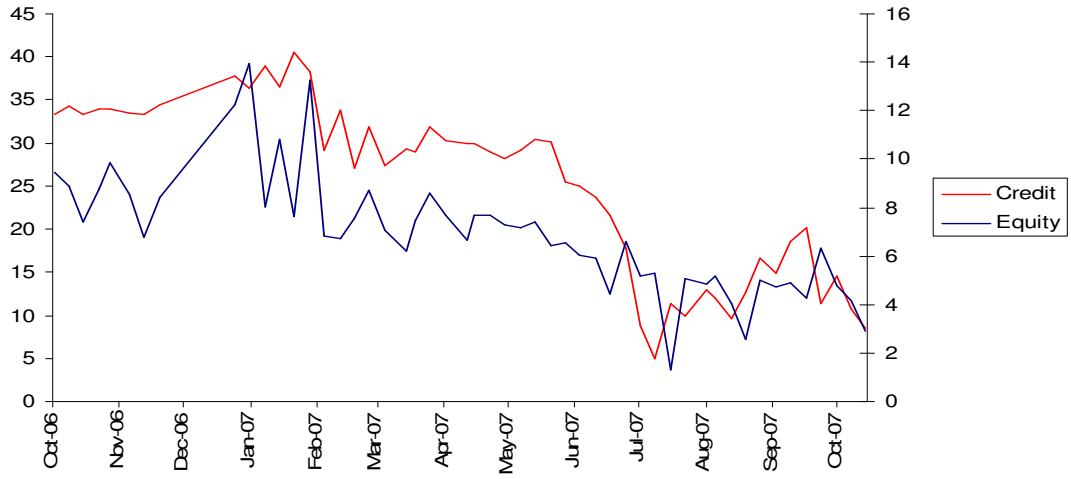
As we have already seen in Figure 10, there are dramatic changes in the skewness and kurtosis of the distribution of expected losses on the CDX after the subprime crisis. In Figure 13 we compare the time-series behaviour of these series with the skewness and kurtosis of the risk-neutral distribution of the equity index (derived from the options which have been extrapolated to a maturity of five years). Movements in skewness and kurtosis across the equity-index and CDX markets are quite close (correlations of 0.81 and 0.80 respectively). The trends are similar, but the equity-index implied values show more week-to-week variation than the CDX-implied values.

Figure 13
Skewness and Kurtosis of the Credit Loss Distribution vs. Implied Systematic Factor's (Equity) Distribution

Panel A: Skewness



Panel B: Kurtosis

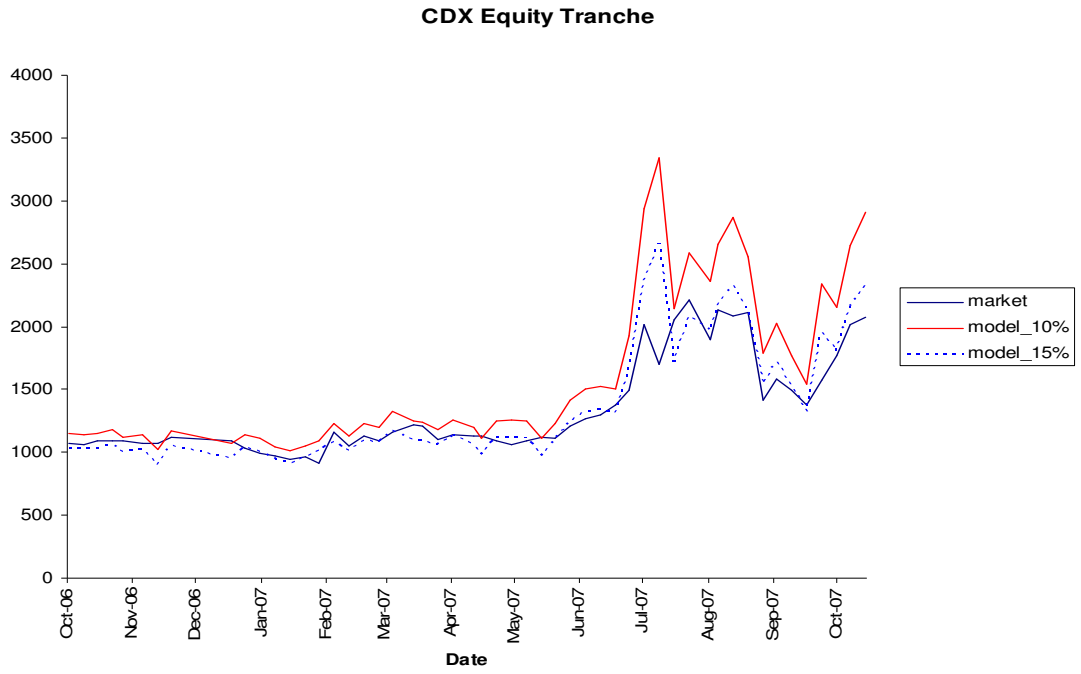


4.3 Equity-implied copula Model with Constant Correlation

The above comparisons give us some confidence that the implied equity-index distribution may be relevant to the pricing of the CDX index (and hence of CDOs on that index), as theory would predict. We now price the equity, mezzanine and senior tranches of CDOs with the equity-implied copula model, which incorporates the equity-index distribution, assuming two different base-correlation levels of 10% and 15%.

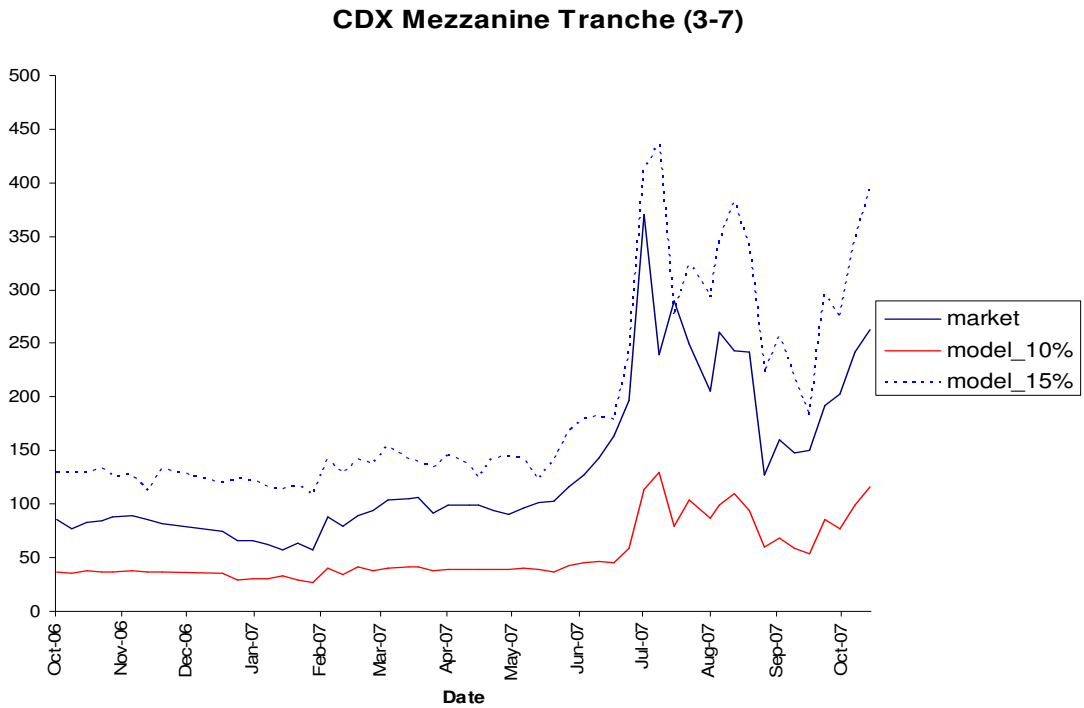
Figure 14A gives the time-series of the spreads on the equity (0-3) tranche in the market, together with the spreads from our equity-implied copula model at 10% and 15% correlations. Until July 2007 the three series move very similarly and give a tranche spread of 1100-1300 basis points. However, after the onset of the subprime crisis the model spreads exceed market spreads, regardless of whether a 10% or a 15% correlation is assumed.

Figure 14A CDX 0-3 Equity-Tranche Spreads: Observed vs. Equity-Implied Copula Model at 10% and 15% correlations



Turning to the mezzanine (3-7) tranche (Figure 14B), before the crisis the market spreads on this tranche are relatively stable, at about 90 basis points, and they lie between the model's spreads computed at 10% correlation (about 40 basis points) and at 15% correlation (about 140 basis points). After the crisis, both model and market spreads rise and the market spread is now quite close to the model one with 15% correlation.

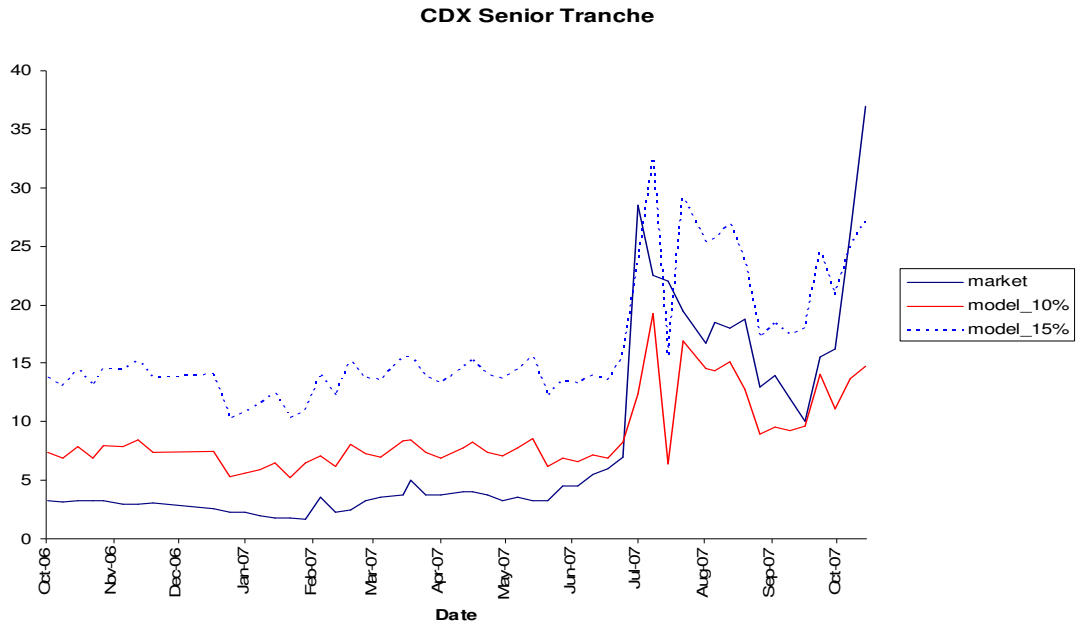
Figure 14B CDX 3-7 Mezzanine Tranche Spreads: Observed vs. Equity-Implied Copula Model at 10% and 15% correlations



Moving to the senior (15-30) tranche (Figure 14C), before the crisis the market spreads lie below the model spreads, even when a low correlation of 10% is assumed in the model. After the crisis there is a huge adjustment in market spreads, from 4 basis point before to about 20 basis points afterwards, and the market spreads mostly lie between the model spreads at 10% and 15% correlations, although there are some very large fluctuations.

Summarising these comparisons of market spreads with equity-implied copula spreads, it is the senior (15-30) tranche for which the model provides quite different results from the market, with a strong indication that senior-tranche spreads were too low before the crisis. There is also some indication that the spread on the equity tranche may have been too small after the crisis, while the middle-risk mezzanine tranche moves approximately in line with model predictions.

Figure 14C CDX 15-30 Senior Tranche Spreads: Observed vs. Equity-Implied Copula Model at 10% and 15% correlations



4.4 Results from Fitting the Equity-Implied Copula to the 0-3 Equity Tranche and then Pricing the 15-30 Senior Tranche at the Same Correlation

We know from the previous section that a base correlation of between 10% and 15% works quite well with the equity-implied copula model in matching spreads on different tranches. However, just as the level of the at-the-money implied volatility on index options changes over time, so the base correlation required to price all of the CDX tranches may also change over time; both depend on traders' expectations. Figure 15A plots the time series of the implied base correlations required by the equity-implied copula model to replicate market spreads for the CDX equity (0-3) tranche. The implied correlation is trending upwards, with a small spike in February 2007 and a large spike in July 2007 (when the crisis started). These two spikes are associated with contemporaneous drops in the equity market, as demonstrated in Figure 15B.

Figure 15A Time Series of the Base Correlations Required to Replicate Market Spreads with the Equity-Implied Copula Model for the 0-3 Equity Tranche,

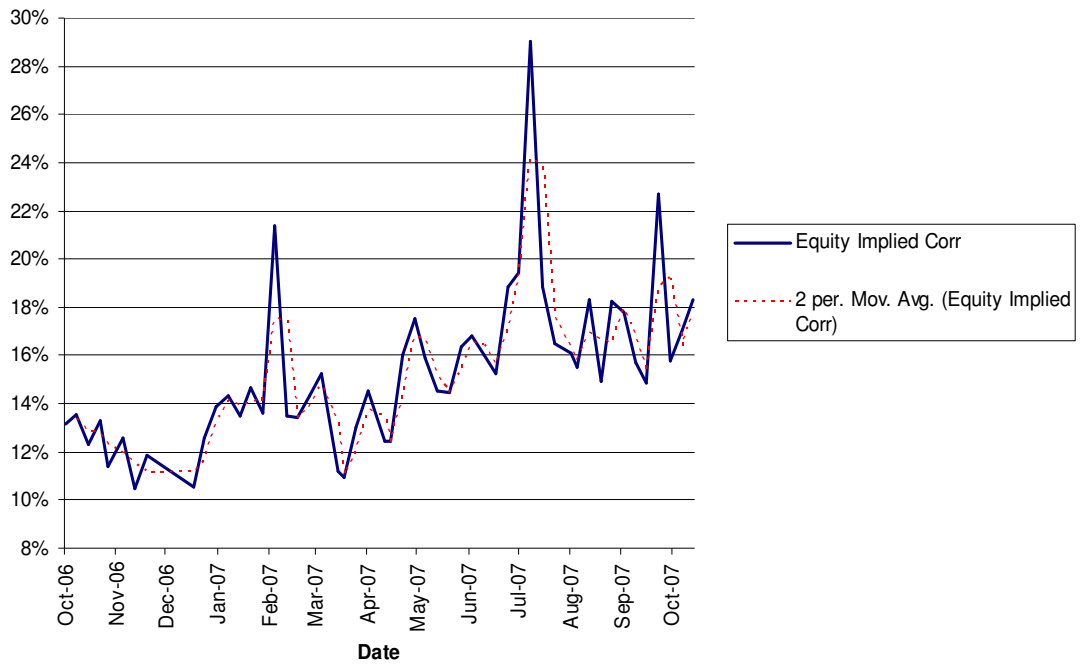


Figure 15B Time Series of Base Correlations (from panel A) vs. S&P 500 Futures Prices

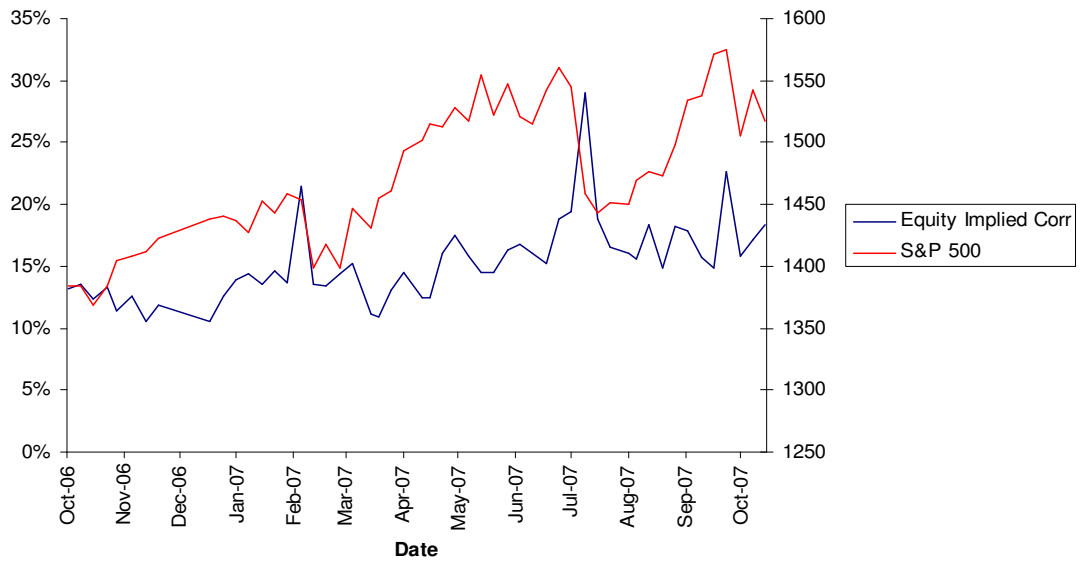


Figure 15C takes the implied correlations for the equity (0-3) tranche on each day in Figure 15A and then uses these correlations with the model to calculate spreads for

the senior (15-30) tranche on the same day. This is a test of the mispricing of the senior tranche relative to the equity tranche. The plots show that the equity-implied copula model (using the base correlation from the equity-tranche) generates spreads on the senior tranche of 10-15 basis points in the period before the subprime crisis, which is much more than the 3-5 basis point spreads in the market at that time. After the onset of the crisis, both model and market senior-tranche spreads increase rapidly, but the model spreads still exceed the market spreads by about 5 basis points.

Figure 15C Time Series of Market Spreads for the Senior 15-30 Tranche vs. Equity-Implied Copula Model Spreads using Equity-Tranche Correlations

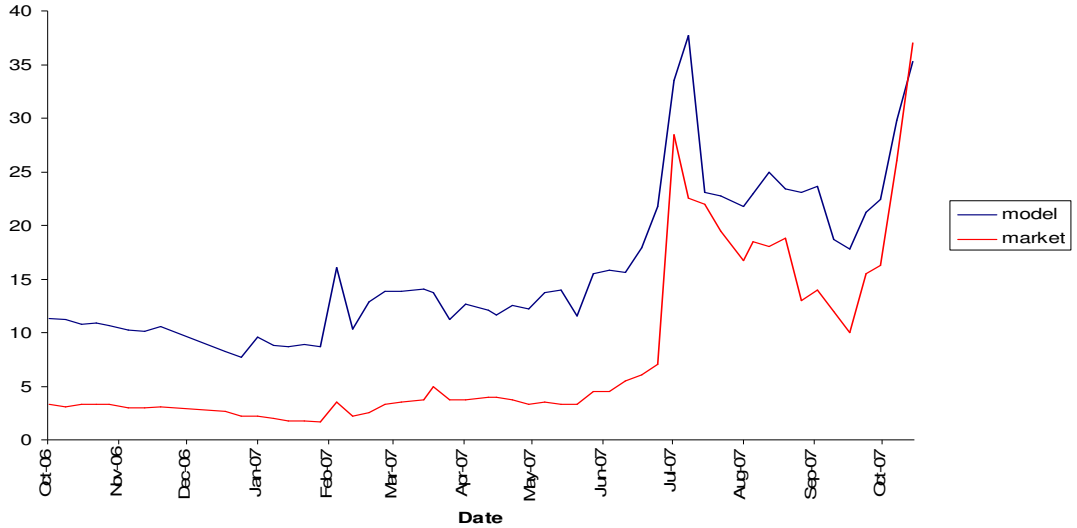
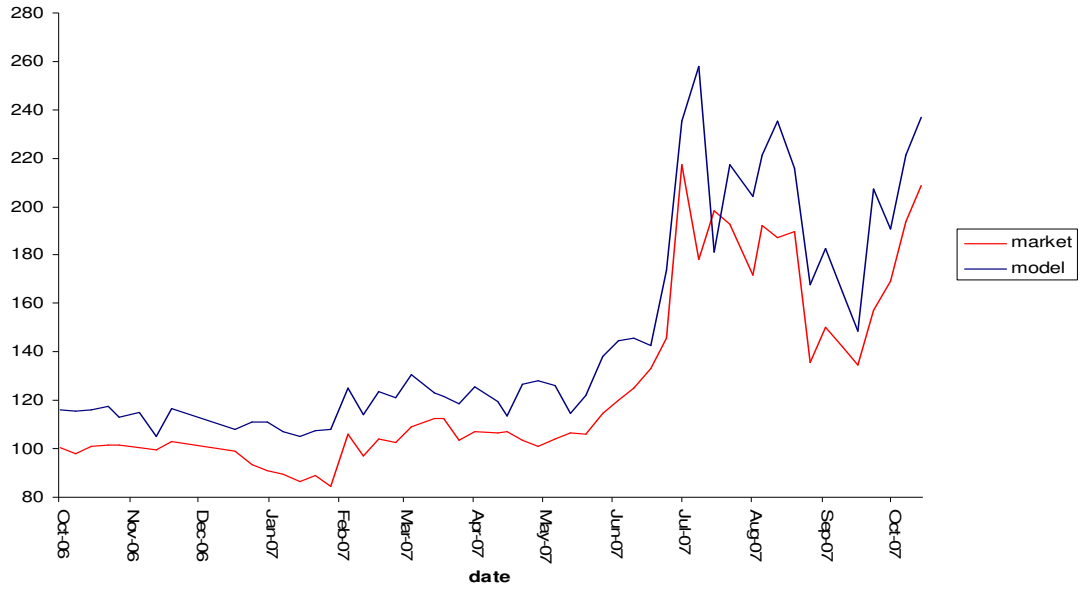


Figure 16 repeats the above analysis, but computes prices for the whole 0-30 first-loss CDX tranche rather than just the senior (15-30) tranche. The model generates spreads on the 0-30 tranche before the crisis of about 120 basis points, as compared with market spreads of about 100 basis points. The difference of 20 basis points remains after the onset of the crisis, despite the jump in market spreads to 200 basis points (on this 0-30 tranche).

Figure 16 Time Series of Equity-Implied Copula Model-Predicted Spreads on 0-30 First-Loss CDO Tranche vs. Market Spreads using Equity-Tranche Correlations



In a similar way (but not plotted here), we find that the spreads from the equity-implied copula model, derived from applying the base correlations on each day estimated from the equity (0-3) tranche, are consistently larger than market spreads for all of the more senior first-loss tranches (0-7, 0-10, 0-15 and 0-30). The average differences are 450, 290, 160 and 20 basis points respectively.

5. Conclusions

The CDX index (of credit-default swaps) depends on the aggregate value of a diversified portfolio of firms. The S&P500 equity index also depends on the value of a diversified portfolio of firms. We therefore develop an equity-implied copula model to price the CDX index and its tranches (CDOs) which is based on the risk-neutral distribution of the S&P500 index, the latter being revealed with equity-index options.

We have first demonstrated that the moments of the distributions implied by equity-index puts and by the CDX index have very similar shapes and move closely together. Then incorporating the implied equity distribution into a copula model for tranches on the CDX (the "equity-implied copula"), we find that model spreads on the senior (15-30) tranche are much larger than those in the market before the subprime crisis. After

the crisis begins, the (15-30) model spreads remain larger, but the difference is much less. Our results are more supportive of the conclusion in Coval et al (2009b) that senior tranches were mispriced before the crisis than of the conclusion of Collin-Dufresne et al (2012) that mispricing was small.

It is easy to look back after the crisis and conclude that the Gaussian copula model used by practitioners for calculating spreads on CDX tranches (i.e. CDOs) was far too simple. However, we show that by making one adjustment, from a Gaussian distribution to a skewed distribution implied from equity-index options, we can make the copula approach far more realistic.

Our results suggest that equity and credit markets are closely integrated, as theory would predict. The same factors appear to generate the puzzles of the large implied base correlation for the senior CDX tranche and of the volatility sneer in equity-index options. The prices of both equity-index options and CDX senior tranches reflect a long-tailed and left-skewed underlying risk-neutral distribution. We think that the most plausible reason for such a skewed distribution is that investors are extremely averse to large downside risks, making them willing to pay a high premium for catastrophe insurance.

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Table 1

Descriptive Statistics for Index and Tranche Spreads on the CDX North American Investment Grade Index.

The numbers in the table relate to credit spreads in basis points. The sample period is October 2006 to November 2007 (57 weekly observations)

CDO Tranche	Mean	Std Dev	Med	Min	Max
Index	45	15	37	30	82
0-3%	1341	384	1134	912	2208
3-7%	135	73	101	58	370
7-10%	35	27	21	10	120
10-15%	17	15	10	4	62
15-30%	8	8	261	2	37

Table 2

Descriptive Statistics for Implied Volatilities for SPX Options Traded on the CBOE

The numbers in the table relate to annualised implied volatilities. The sample period is October 2006 to November 2007 (57 weekly observations). The forward put moneyness and time to maturity are denoted in brackets, e.g. (0.7k, 1m) denotes 0.7 forward put moneyness and 1 month time to maturity.

Moneyness and Maturity (months)	Mean	Std Dev	Med	Min	Max
(0.7k, 1m)	29.5%	3.7%	28.6%	24.5%	38.7%
(0.7k, 3m)	23.2%	3.7%	22.0%	18.7%	32.1%
(0.7k, 6m)	20.8%	3.7%	19.3%	16.4%	29.8%
(0.9k, 1m)	19.2%	3.8%	17.6%	14.9%	28.4%
(0.9k, 3m)	17.5%	3.9%	15.9%	13.3%	26.9%
(0.9k, 5m)	16.9%	3.9%	27.9%	12.7%	26.4%

Table 3
Comparison between the CDO Tranche Spreads Calculated with the Equity-Implied Copula Model and with the Gaussian Copula Model under the Same Parameter Settings (portfolio average default probability of 5%, asset correlation of 25%, with index-option data from October 27th 2006)

CDO Tranche	Gaussian Copula Spread (bp)	Implied-Equity Copula Spread (bp)
0-3%	1658	1012
3-7%	417	243
7-10%	155	146
10-15%	61	97
15-30%	9	42

Table 4

Implied Base Correlations Required by the Gaussian Copula to Give Spreads Generated by the Equity-Implied Copula under the Same Parameter Settings
(portfolio average default probability of 5%, asset correlation of 25%, with index-option data from October 27th 2006)

CDO First-Loss Tranche	Implied Base Correlation with Gaussian-Copula
0-3%	45.43%
0-7%	49.82%
0-10%	52.64%
0-15%	55.72%
0-30%	61.30%