Derivatives Pricing under a New Macro-financial Square-root Process for the Term Structure of Interest Rates

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Abstract

This paper presents a new macro-financial continuous-time model for the term structure of interest rates assuming that the spot rate converges to a certain long-term level that depends on the business cycle. In addition, we consider that the interest rate volatility depends on the interest rate level. In short, both the mean reversion level and the interest rate volatility are modeled by using a harmonic oscillator. Under these assumptions, we compute closed-form expressions for the values of different fixed income and interest rate derivatives and for relevant risk management measures.

Keywords: square-root process, interest rates, term structure, continuous-time model, harmonic waves, martingale.

JEL classification: G12, G13.

I Introduction

Through the time, modeling the term structure of interest rates has been the object of many studies and the aim of attention for economists and financial institutions. Models proposed in the academic literature can be classified in endogenous and exogenous.

Endogenous models make certain assumptions on the factors that drive the term structure and their stochastic processes. The term structure is fully characterized by these endogenous factors meaning that the current term structure of rates is an output rather than an input of the model. Examples of one-factor models are Vasicek (1977), Brennan and Schwartz (1980), or Cox *et al.* (1985) (CIR from now on). The main drawback of these models is the lack of empirical realism. Consequently, these models do not fit accurately the current term structure and, then, they do not price correctly. In order to cope with this problem, we can find two-factor models such as Schaefer and Schwartz (1984), Longstaff and Schwartz (1992), or Duffie and Kan (1996) and Chen (1996) who introduced a three-factor model.

On the other hand, exogenous models take the current term structure as given and derive future changes in interest rates avoiding intertemporal arbitrage opportunities. The first contribution was made by Ho and Lee (1986) who showed how to build a model consistent with the initial term structure of interest rates. Since this model has some disadvantages, their work has been extended by a number of authors such as Black *et al.* (1990), Hull and White (1990, 1993), Black and Karasinski (1991), Heath *et al.* (1992), and Mercurio and Moraleda (2000). We can mention the models introduced in Hull and White (1990, 1993) that extend those presented in Vasicek (1977) and Cox *et al.* (1985) by letting the parameters be time-dependent and adding more flexibility to fit the initial term structure

of interest rates. For a thoroughly literature review on term structure models refer to, for instance, Webber and James (2001), Brigo and Mercurio (2006), or Filipović (2009).

In this paper, we propose a new one-factor macro-financial square-root model where the instantaneous interest rate is pulled back to a certain long term level that follows a cyclic behavior. Under this assumption, the whole term structure of interest rates is fully determined by the instantaneous spot rate.

The intuitive idea behind this model is that interest rates converge to a lower level during economic recessions, while this reversion level tends to be higher during economic expansions. We can graphically illustrate this idea in Figure 1, which shows the effective Federal funds rates from 1955 to 2010 and indicates the recessions happening during this period.

[INSERT FIGURE 1 AROUND HERE]

In order to reflect this cyclic behavior, we model the mean reversion level using a harmonic oscillator. Moreover, empirical evidence (see, for instance, Chan *et al.* (1992)) illustrated that interest rate volatility depends on the interest rates level. Hence, we model interest rate volatility using a similar functional form. Under these assumptions, we analytically price zero-coupon bonds and different derivatives such as forward on bonds, European options on zero-coupon and coupon-bearing bonds, European bond forward options, swaps, swaptions, caps, floors, collars, and provide some risk management measures.

This paper is organized as follows. In Section 2, we introduce the new model and its practical implications. Section 3 presents the general pricing partial differential equation and derives closed-form expressions for different derivatives. Finally, Section 4 summarizes the main findings and conclusions. Mathematical proofs are deferred to the Appendix.

II A New Square-root Model for the Term Structure

In this section, we propose our model, the specific functional form for each time-dependent parameter, and describe all the practical implications arising from this model.

Unlike many other one-factor models that allow the spot rate process for time-dependent parameters (see, for instance, Hull and White (1990, 1993)), we now assume that the mean reversion level depends on the business cycle. We also consider that the interest rate volatility depends on the interest rate level. In order to model the behavior of both variables, we assume a harmonic oscillator given as

$$f(t) = A\sin(\varphi - wt)$$

where A denotes the amplitude of the wave, φ the offset phase, and w the temporal frequency.

We now define the mean reversion level, θ_t, and the volatility, σ_t^2, as

$$\theta_t = A_\theta \sin^2(\varphi - wt) \tag{1}$$

$$\sigma_t^2 = A_\sigma \sin^2(\varphi - wt) \tag{2}$$

Hence, the positiveness of the mean reversion level and the interest rate volatility is guaranteed.

Let r_t denote the instantaneous interest rate available at time t whose dynamic is

$$dr_t = \mu_r dt + \sigma_r dW_t \tag{3}$$

where W_t is a standard Wiener process and

$$\mu_r = k(\theta_t - r_t), k \in R^+, \quad \sigma_r = \sigma_t\sqrt{r_t} \tag{4}$$

Looking at these expressions, it is clear that our model nests that presented in Cox *et al.* (1985) making $w = 0$ in equations (1)-(2).

For square-root processes of this type, Cox *et al.* (1985) shows that the distribution function of interest rates is a rescaled non-central chi-square with δ degrees of freedom. Note that, whenever δ is not a positive integer, the distribution of r_t is unknown. Besides, the dimension of the process r_t is given by $\delta = \frac{4\theta_t k}{\sigma_t^2}$. As both waves are in phase, the model's dimension can be represented as $\delta = \frac{4A_\theta k}{A_\sigma} > 0$.[1]

Our model guarantees the positiveness of interest rates. On this respect, Feller (1951) studied the Fokker-Plank-Kolmogorov equation for the transition density and showed that $r_t > 0$ if $\delta \geq 2$, however it can become null if $\delta < 2$ but will never become negative.

III Pricing

This section presents closed-form expressions for the price of zero-coupon bonds and, later, we analytically compute closed-form formulas for the prices of different securities.

Let $P(r_t, t, T)$ denote the price at time t of a zero-coupon bond that pays \$1 at maturity T. Then, the bond price dynamics is given by the process

$$dP = \mu_P(r_t, t, T)P(r_t, t, T)dt + \sigma_P(r_t, t, T)P(r_t, t, T)dW_t \tag{5}$$

Applying Itô's Lemma and using (3), it can be shown that

$$\mu_P = \frac{1}{P}\left(P_t + \mu_r P_r + \frac{1}{2}\sigma_r^2 P_{rr}\right) \tag{6}$$

$$\sigma_P = \sigma_r \frac{P_r}{P} \tag{7}$$

[1]Note that, if $\sin(\varphi - wt)$ is equal to zero, then δ becomes indeterminate. As this case would only occur for a infinitesimal period of time, we do not consider this possibility.

where arguments have been omitted and subscripts in P indicate the corresponding partial derivative. Applying standard no-arbitrage arguments, there exists a factor $\Lambda(r_t, t)$, called market price of risk, such that

$$\mu_P(r_t, t, T) - r_t = \Lambda(r_t, t)\sigma_P(r_t, t, T) \tag{8}$$

Finally, some trivial algebra leads to the following partial differential equation (PDE)

$$P_t(r_t, t, T) + (\mu_r - \Lambda(r_t, t)\sigma_r)P_r(r_t, t, T) + \frac{1}{2}\sigma_r^2 P_{rr}(r_t, t, T) - r_t P(r_t, t, T) = 0 \tag{9}$$

that must be verified by the price of any derivative.

A Bond Pricing

We consider a market price of risk such as

$$\Lambda(r_t, t) = \frac{\lambda_t \sqrt{r_t}}{\sigma_t} \tag{10}$$

Using expressions (4)-(10), the PDE (9) becomes

$$P_t(r_t, t, T) + (k(\theta_t - r_t) - \lambda_t r_t)P_r(r_t, t, T) + \frac{1}{2}\sigma_t^2 r_t P_{rr}(r_t, t, T) - r_t P(r_t, t, T) = 0 \tag{11}$$

The solution of this PDE, subject to the boundary condition $P(r_T, T, T) = 1,\ \forall r_T$, is given by the following Proposition.

Proposition 1 *The price at time t of a zero-coupon bond with maturity T and \$1 face value is given by*

$$P(r_t, \tau) = A(\tau)e^{-B(\tau)r_t}$$

where

$$
\begin{aligned}
A(\tau) &= \exp\left\{\int_t^T k\theta_t B(\tau)dt\right\} \\
B(\tau) &= \frac{c_1 MC(a,q,x) + MS(a,q,x)}{\frac{1}{2}(\lambda+k)\left(c_1 MC(a,q,x) + MS(a,q,x)\right) + w\left(c_1 MCP(a,q,x) + MSP(a,q,x)\right)} \\
a &= -\frac{A_\sigma + (\lambda+k)^2}{4w^2} \\
q &= -\frac{A_\sigma}{8w^2} \\
x &= \varphi - wt \\
c_1 &= -\frac{MS(a,q,\varphi - wT)}{MC(a,q,\varphi - wT)} \\
\tau &= T - t
\end{aligned}
$$

where θ_t is given by (1), MC and MS represent the Mathieu cosine and sine function, respectively, and MCP and MSP represent the derivative with respect to x of the Mathieu cosine and sine function, respectively. ■

Proof. See the Appendix ■

Figure 2 compares the bond price in the CIR model against three alternatives in our model. We check that, in our model, the bond price does not decrease monotonically with time to maturity. Additionally, we provide much more flexibility than the CIR model with the same analytical tractability. We can also visualize the presence of humps, which is a very desirable effect not only here but also in any interest rate derivative.

[INSERT FIGURE 2 AROUND HERE]

The following Corollary immediately arises.

Corollary 1 *As a coupon bond can be interpreted as a portfolio of zero-coupon bonds, pricing of coupon bonds is straightforward applying Proposition 1.* ■

Replacing the bond price expression obtained in Proposition 1 into (6)-(7) and using (4)-(10), we get the next Corollary.

Corollary 2 *The bond price dynamics under the no-arbitrage condition is given as*

$$dP(r_t, t, T) = \mu_P(r_t, t, T)P(r_t, t, T)dt + \sigma_P(r_t, t, T)P(r_t, t, T)dW_t$$

where

$$\mu_P(r_t, t, T) = r_t(1 - \lambda_t B), \quad \sigma_P(r_t, t, T) = -B\sigma_t\sqrt{r_t}$$

■

Note that, in a risk-neutral world, where $\lambda_t = 0$, the bond price process is a martingale. Under this framework, the term structure is fully characterized considering the zero-coupon bond price $P(r_t, t, T)$ given by Proposition 1, as stated in the following Corollary.

Corollary 3 *The yield to maturity, $R(r_t, t, T)$, is given by*

$$R(r_t, t, T) = -\frac{1}{\tau}\ln P(r_t, t, T), \ \tau = T - t$$

The short-term interest rate is defined as the instantaneous interest rate at time t, that is,

$$r_t = \lim_{\tau \to 0} R(r_t, t, T) = R(r_t, t, t)$$

The instantaneous forward rate is given as

$$f(r_t, t, T) = -\frac{\partial \ln(P(r_t, t, T))}{\partial T}$$

■

Figure 3 shows the term structure of interest rates in the CIR model and three alternatives in our model. We can see how our model adds flexibility as we can reflect different behaviors for the term structure.

[INSERT FIGURE 3 AROUND HERE]

For illustrative purposes, Figures 4 and 5 show how the term structure of interest rates responds to changes in the mean reversion speed and volatility in both models. In the CIR model, the higher the speed of mean reversion, the higher the interest rate while, in our model, the lower the mean reversion speed, the flatter the term structure. Besides, in our model, there is a twist in the pattern due to the cyclic behavior. In Figure 5, for both models, the higher the volatility, the lower the term structure.

[INSERT FIGURES 4-5 AROUND HERE]

Figures 6 and 7 reflect the response of the term structure of interest rates to different values of the mean reversion level in both models. In the CIR model, the higher the mean reversion level, the higher the yield. In our model, it is harder to analyze this response as it depends on three parameters. Anyway, we observe that the lower the amplitude, the flatter and the lower the term structure. When changing the temporal frequency, it seems clear that the higher the temporal frequency, the more humped the term structure. Finally, for different offset phases, the curves occasionally crossover each other.

[INSERT FIGURES 6-7 AROUND HERE]

On the risk management side, we get the following corollary.

Corollary 4 *The two major bond risk measures, duration and convexity, are given as*

- *Duration measures the bond price sensitivity for a change in interest rates:*

$$Duration = -\frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial r_t} = B(t,T)$$

- *Convexity measures how duration changes with interest rates:*

$$Convexity = \frac{1}{P(t,T)}\frac{\partial^2 P(t,T)}{\partial r_t^2} = B^2(t,T)$$

with $B(t,T)$ as given by Proposition 1. ■

B Pricing of Bond Derivatives

The PDE (11) subject to the appropriate boundary condition will lead us to value any interest rate derivative. Consider a derivative whose pay-off at time s is given by $U_s(r_s)$.[2] Applying the fundamental results of Heath *et al.* (1992), there exists an unique equivalent risk-neutral measure $\widetilde{P}$ such that the value at time t of this derivative U_t can be represented as

$$U_t(r_t,t,s) = \widetilde{E}\left[e^{-\int_t^s r_u du} U_s(r_s)|r_t\right]$$

where $\widetilde{E}$ denotes expectation with respect to the risk-neutral measure $\widetilde{P}$.

Let

$$\widetilde{W_t} = W_t + \int_0^t \Lambda_u du \tag{12}$$

denote the Wiener process under $\widetilde{P}$. Under this measure, applying (5), (8), and (12), the risk-neutral dynamics of the bond price is given as

$$dP(r_t,t,s) = r_t P(r_t,t,s)dt + \sigma_P P(r_t,t,s)d\widetilde{W_t}$$

[2]Obviously, if $U_s(r_s) = 1$, the previous bond price expression is obtained.

Hence, the discount bond price process is a martingale.

Define $g(P(r_t,t,s)) = \ln P(r_t,t,s)$. Then, applying Itô's Lemma, we get

$$dg_t = \frac{\partial g(P)}{\partial P}dP - \frac{1}{2}\frac{\partial^2 g(P)}{\partial P^2}(dP)^2 = \frac{1}{P}\left(r_t P dt + \sigma_P P d\widetilde{W}_t\right) - \frac{1}{2}\frac{1}{P^2}\sigma_P^2 P^2 dt$$

where the corresponding arguments have been omitted. Integrating from 0 to t, we get

$$P(r_t,t,s) = P(r_0,0,s)\exp\left(\int_0^t r_u du + \int_0^t \sigma_P(u,s)d\widetilde{W}_u - \frac{1}{2}\int_0^t \sigma_P^2(u,s)du\right)$$

Hence, for each $s \in [t,T]$, the process Z_t^s defined as

$$Z_t^s = \frac{P(r_t,t,s)}{P(r_0,0,s)}e^{-\int_0^t r_u du} = \exp\left(\int_0^t \sigma_P(u,s)d\widetilde{W}_u - \frac{1}{2}\int_0^t \sigma_P^2(u,s)du\right)$$

is a martingale. Moreover, in line with Karatzas and Shreve (1988), we get

$$Z_t^s = 1 + \int_0^t \sigma_P(u,s)\exp\left(\int_0^t \sigma_P(x,s)d\widetilde{W}_x - \frac{1}{2}\int_0^t \sigma_P^2(x,s)dx\right)d\widetilde{W}_u$$

Thus, by the Girsanov's theorem, for each $s \in [t,T]$, there exists an equivalent s-forward measure P^s such that

$$W_t^s = \widetilde{W}_t + \int_0^t \sigma_P(u,s)du \tag{13}$$

where W_t^s represent a standard Wiener process under P^s.

Karatzas and Shreve (1988) shows that, for a random variable Y

$$E^s\left[Y|F_t\right] = \frac{1}{Z_t^t}E\left[Z_t^s Y|F_t\right]$$

Then, the following Proposition presents the equivalent change of measure.

Proposition 2 *Under $\widetilde{P}$, the value at time t of any derivative $U_t(r_t,t,s)$ given by*

$$U_t(r_t,t,s) = \widetilde{E}\left[e^{-\int_t^s r_u du}U_s(r_s)|r_t\right]$$

has an equivalent s-forward measure, P^s, such that

$$U_t(r_t,t,s) = P(r_t,t,s)\widetilde{E}\left[\frac{Z_t^s}{Z_t^t}U_s(r_s)|r_t\right] = P(r_t,t,s)E^s\left[U_s(r_s)|r_t\right]$$

where E^s represents expectation under P^s and

$$\frac{Z_t^s}{Z_t^t} = \frac{e^{-\int_t^s r_u du}}{P(r_t,t,s)}$$

■

Moreover, the instantaneous forward rate is given as

$$f(r_t,t,T) = D(t,T) + B_T(t,T)r_t \tag{14}$$

with

$$D(t,T) = -\frac{A_T(t,T)}{A(t,T)} = \frac{\delta}{4}\int_t^T B_T(u,T)\sigma_u^2 du \tag{15}$$

Applying Itô's Lemma and using the spot rate dynamics (3), we get

$$df(r_t,t,T) = \mu_f dt + \sigma_f dW_t \tag{16}$$

where

$$\mu_f = \left(B_{Tt}(t,T) - kB_T(t,T)\right)r_t, \quad \sigma_f = B_T(t,T)\sigma_t\sqrt{r_t}$$

Similarly to Heath *et al.* (1992), the following restriction on the forward rate drift is verified

$$\mu_f(\omega,t,T) = \sigma_f(\omega,t,T)\left[\int_t^T \sigma_f(\omega,t,x)dx + \Lambda_t\right]$$

Now, replacing (13) into the forward rate process (16), we get

$$df(r_t,t,s) = \sigma_f\left[\int_t^s \sigma_f(t,v)dv + \Lambda_t\right]dt + \sigma_f\left(dW_t^s - \Lambda_t dt - \sigma_P dt\right)$$

Hence, using $\int_t^s \sigma_f(r_t, t, v)dv = \sigma_P(r_t, t, s)$ leads to

$$df(r_t, t, s) = \sigma_f(r_t, t, s)dW_t^s = B_s(t, s)\sigma_t\sqrt{r_t}dW_t^s \tag{17}$$

Then, comparing equations (16) and (17), we get

$$dr_t = k^s\left(\theta_t^s - r_t\right)dt + \sigma_t\sqrt{r_t}dW_t^s$$

where

$$\begin{aligned} k^s &= \frac{B_{st}(t, s)}{B_s(t, s)} \\ \theta_t^s &= \frac{k}{k^s}\theta_t \end{aligned}$$

Hence, under the s-forward measure, the instantaneous interest rate follows a CIR-type process with speed and level of mean reversion given by k^s and θ_t^s, respectively. Then, standard methods applied in Cox *et al.* (1985) can be used.

Define the state variable $X^s(t) = (x_1^s(t), ..., x_d^s(t))$ as the process generating $r_s = \| X^s \|^2$. The state variable dynamics for $x_i^s(t)$ is given by

$$dx_i^s(t) = \frac{1}{2}\sigma_t\sqrt{B_s(t, s)}dW_i^s(t)$$

Hence,

$$\begin{aligned} r_s &= \sum_{i=1}^{d}\left[\frac{1}{2}\int_t^s \sigma_u\sqrt{B_s(u, s)}dW_i^s(u) + x_i^s(u)\right]^2 \\ &= \frac{d}{4}\int_t^s \sigma_u^2 B_s(u, s)du + \| X^s(t) \| \int_t^s \sigma_u\sqrt{B_s(u, s)}dW^s(u) + \| X^s(t) \|^2 \end{aligned} \tag{18}$$

where $W^s(t) = (W_1^s(t), ..., W_d^s(t))$ is a d-dimensional Wiener process under P^s.

Note that, under P^s, the instantaneous forward rate can be represented as

$$f(t, s) = \| X^s(t) \|^2 + D(t, s) \tag{19}$$

Taking conditional expectations under P^s in (18) and using (14)-(15) and (19), we get

$$E^s\left[r_s|r_t\right] = D(t,s) + B_s(t,s)r_t$$

Defining $d = \delta = \frac{4\theta_t k}{\sigma_t^2}$ and $\varpi = \frac{\delta}{D(t,s)}r_s$, we obtain

$$E^s\left[\varpi|r_t\right] = \delta + \frac{\delta B_s(t,s)}{D(t,s)}r_t$$

Hence, ϖ follows a non-central chi-square distribution with δ degrees of freedom and non-centrality parameter $\left(\delta B_s(t,s)/D(t,s)\right)r_t$.

The value at time t of a derivative whose pay-off at time s is given by $U_s(r_s)$ will read as

$$U_t(r_t,t,s) = \widetilde{E}\left[e^{-\int_t^s r_u du}U_s(r_s)|r_t\right]$$

Applying Proposition 2, the next Proposition arises.

Proposition 3 *The value at time t of any interest rate derivative with terminal pay-off $U_s(r_s)$ is given by*

$$U_t(r_t,t,s) = P(r_t,t,s)E^s\left[U_s(\varpi)|r_t\right]$$

where $P(r_t,t,s)$ is given by Proposition 1, E^s represents expectation under P^s, and

$$\varpi = \frac{\delta}{D(t,s)}r_s \sim \chi^2\left(\delta, \frac{\delta B_s(t,s)}{D(t,s)}r_t\right) \tag{20}$$

with $B(s,T)$ as given by Proposition 1 and $D(t,s)$ as given by (15). ■

After obtaining this general closed-form expression, we analyze several particular cases:

1. **Forward on zero-coupon bond**

Consider a forward contract expiring at time s written on a zero-coupon bond maturing at time $T > s$ and \$1 face value. Then, under the s-forward measure P^s, the delivery price established at time t for this forward contract is given as

$$F(r_t, t, s, T) = E^s \left[P(r_s, s, T) | r_t \right]$$

Then, using Proposition 3, the value of this bond forward is given as follows.

Proposition 4 *The value at time t of a bond forward maturing at time s, written on a zero-coupon bond expiring at time T and \$1 face value is given by*

$$\begin{aligned} F(r_t, t, s, T) &= E^s \left[P(r_s, s, T) | r_t \right] \\ &= \frac{A(s,T) e^{-\frac{1}{2}(\xi_1 - \xi_2)}}{\left(\frac{2}{\delta} B(s,T) D(t,s) + 1 \right)^{\frac{\delta}{2}}} \end{aligned}$$

where $A(s,T)$ and $B(s,T)$ are given by Proposition 1, $D(t,s)$ as given by (15), and

$$\begin{aligned} \xi_1 &= \frac{\delta B_s(t,s)}{D(t,s)} r_t \\ \xi_2 &= \frac{\xi_1}{\frac{2}{\delta} B(s,T) D(t,s) + 1} \end{aligned}$$

■

2. **European option on zero-coupon bond**

Consider a European call option maturing at time s with strike K, written on a zero-coupon bond that matures at time $T > s$. Let $c_t(r_t, s, T, K)$ denote the price at time t of an European call option. Then, the boundary condition of the PDE (11) is given by

$$c_s(r_s, s, T, K) = \max\{P(r_s, s, T) - K, 0\}$$

Under the risk-neutral measure $\widetilde{P}$, the price at time t of this call option is given by

$$c_t(r_t, s, T, K) = \widetilde{E}\left[e^{-\int_t^s r_u du}\left(P(r_s, s, T) - K\right)^+ |r_t\right]$$

Then, using Proposition 3, we price this option as follows.

Proposition 5 *The price at time t of a European call option with maturity s written on a zero-coupon bond expiring at time T and \$1 face value is given by*

$$\begin{aligned} c_t(r_t, s, T, K) &= P(r_t, t, s)E^s\left[\left(A(s,T)e^{-B(s,T)r_s} - K\right)^+ |r_t\right] \\ &= P(r_t, t, s)F(r_t, t, s, T)\chi^2\left(\rho_2, \delta, \xi_2\right) - P(r_t, t, s)K\chi^2\left(\rho_1, \delta, \xi_1\right) \end{aligned}$$

where $\chi^2(\cdot)$ denotes the non-central chi-square distribution function and

$$\begin{aligned} \rho_1 &= \frac{\delta}{B(s,T)D(t,s)}\ln\left(\frac{A(s,T)}{K}\right) \\ \rho_2 &= \rho_1\left(\frac{2}{\delta}B(s,T)D(t,s) + 1\right) \\ \xi_1 &= \frac{\delta B_s(t,s)}{D(t,s)}r_t \\ \xi_2 &= \frac{\rho_1}{\rho_2}\xi_1 \end{aligned}$$

with $P(r_t, t, s)$, $A(\cdot,\cdot)$, and $B(\cdot,\cdot)$ as given in Proposition 1, $F(r_t, t, s, T)$ as given by Proposition 4, and $D(t, s)$ as given by equation (15). ■

Proof. See the Appendix. ■

3. **European option on coupon bond**

Consider a European call option that matures at time s and strike K. The underlying asset is a coupon bond maturing at time T paying N coupons α_i at times j_i, $i =$

$1, 2, \cdots, N$ where $j_1 > s,\ j_N = T$. The price of this coupon bond at time s is given as the sum of the corresponding zero-coupon bonds, that is,

$$P(r_s, s, T) = \sum_{i=1}^{N} \alpha_i P(r_s, s, j_i) \tag{21}$$

where $P(r_s, s, j_i),\ i = 1, 2 \cdots, N$ is given by Proposition 1.

Let $c_t\left(r_t, s, \{j_i\}_{i=1}^N, K\right)$ denote the price at time t of this call option. Using (21), the boundary condition of the PDE (11) becomes now

$$c_s\left(r_s, s, \{j_i\}_{i=1}^N, K\right) = \max\left\{\sum_{i=1}^{N} \alpha_i P(r_s, s, j_i) - K, 0\right\}$$

Applying Proposition 3, the call option price is given as

$$\begin{aligned} c_t\left(r_t, s, \{j_i\}_{i=1}^N, K\right) &= P(r_t, t, s) E^s\left[c_s\left(\varpi, s, \{j_i\}_{i=1}^N, K\right) | r_t\right] \\ &= P(r_t, t, s) E^s\left[\max\left\{\sum_{i=1}^{N} \alpha_i P(\varpi, s, j_i) - K, 0\right\} | r_t\right] \end{aligned}$$

In line with Jamshidian (1989), we find $K_i,\ i = 1, 2, \cdots, N$ such that

$$\max\left\{\sum_{i=1}^{N} \alpha_i P(\varpi, s, j_i) - K, 0\right\} = \sum_{i=1}^{N} \alpha_i \max\left\{P(\varpi, s, j_i) - K_i, 0\right\} \tag{22}$$

where $K_i = P(\kappa, s, j_i)$ and κ is the solution of $\sum_{i=1}^N \alpha_i P(\kappa, s, j_i) = K$.[3]

Hence, this option can be interpreted as a portfolio of European call options on zero-coupon bonds with "appropriate" strikes K_i as stated in the following Proposition.

Proposition 6 *The price at time t of a European call option with maturity s on a coupon bond expiring at T, paying coupons α_i at times $j_i,\ i = 1, 2, \cdots, N$ is given by*

$$c_t\left(r_t, s, \{j_i\}_{i=1}^N, K\right) = \sum_{i=1}^{N} \alpha_i\ c_t\left(r_t, s, j_i, K_i\right)$$

[3]Note that the existence of strikes K_i such that (22) has a solution is guaranteed as the bond price decreases with the instantaneous interest rate.

where $c_t(r_t, s, j_i, K_i)$ is given by Proposition 5. ■

4. **European bond forward option**

Consider a European bond forward call option maturing at time s with strike K, where the underlying asset is a bond forward contract with expiration date T_f written on a zero-coupon bond that matures at time $T_b > T_f > s$ and \$1 face value.

Let $c_t(r_t, s, T_f, T_b, K)$ denote the price at time t of this call option.

Then, the boundary condition for the PDE (11) is given by

$$c_s(r_s, s, T_f, T_b, K) = \max\{F(r_s, s, T_f, T_b) - K, 0\}$$

Under the risk-neutral measure $\widetilde{P}$, the price at time t of this option is given as

$$c_t(r_t, s, T_f, T_b, K) = \widetilde{E}\left[e^{-\int_t^s r_u du}\left(F(r_s, s, T_f, T_b) - K\right)^+ |r_t\right]$$

Applying Proposition 3, the price of this option is given by the following Proposition.

Proposition 7 *The price at time t of a European bond forward call option that matures at time s on a forward contract expiring at time T_f written on a zero-coupon bond maturing at time T_b and \$1 face value is given by*

$$\begin{aligned} c_t(r_t, s, T_f, T_b, K) &= P(r_t, t, s)E^s\left[(F(r_s, s, T_f, T_b) - K)^+ | r_t\right] \\ &= P(r_t, t, s)\Theta(r_t, t, s, T_f, T_b)\chi^2(\rho_2, \delta, \xi_2) - P(r_t, t, s)K\chi^2(\rho_1, \delta, \xi_1) \end{aligned}$$

where $\chi^2(\cdot)$ denotes the non-central chi-square distribution function and

$$\Theta(r_t, t, s, T_f, T_b) = \frac{A(T_f, T_b)e^{-\frac{1}{2}(\xi_1 - \xi_2)}}{\left(\left(\frac{2}{\delta}B(T_f, T_b)D(s, T_f) + 1\right)\left(\frac{D(t,s)}{\delta}H(s, T_f, T_b) + 1\right)\right)^{\frac{\delta}{2}}}$$

$$
\begin{aligned}
\rho_1 &= \frac{2\delta \ln\left(\frac{A(T_f,T_b)}{K\left(\frac{2}{\delta}B(T_f,T_b)D(s,T_f)+1\right)^{\frac{\delta}{2}}}\right)}{H(s,T_f,T_b)D(t,s)} \\
\rho_2 &= \rho_1\left(\frac{D(t,s)}{\delta}H(s,T_f,T_b)+1\right) \\
\xi_1 &= \frac{\delta B_s(t,s)}{D(t,s)}r_t \\
\xi_2 &= \frac{\rho_1}{\rho_2}\xi_1 \\
H(s,T_f,T_b) &= \frac{2\delta B(T_f,T_b)B_{T_f}(s,T_f)}{2B(T_f,T_b)D(s,T_f)+\delta}
\end{aligned}
$$

where $P(r_t,t,s)$, $A(\cdot,\cdot)$, and $B(\cdot,\cdot)$ as given in Proposition 1 and $D(\cdot,\cdot)$ as in (15). ■

Remark 1 *Note that $P(r_t,t,s)\Theta(r_t,t,s,T_f,T_b)$ can be interpreted not as the forward price at time t, but as the price at time t of an asset paying the forward price at time s.* ■

Corollary 5 *For all the above cases, put option prices arise directly from put-call parity.* ■

C Interest Rate Derivatives

In this subsection we focus our attention on pricing "pure" interest rate derivatives, that is, derivatives whose underlying is directly the interest rate. We start pricing FRA's, and then, we move to more complicated products such as swaps, caps, floors, and collars.

1. **Forward Rate Agreement**

   Consider a FRA with \$1 notional value and maturity s, where the investor agrees to pay a fixed interest rate K and receive a floating rate with tenor $T_s - s$. The floating rate is set at time s and the net cash-flow is received at time $T_r > s$.

   Then, under the risk-neutral measure $\widetilde{P}$, the FRA value at time t is given by

$$
FRA_t(r_t,s,T_r,T_s,K) = \widetilde{E}\left[e^{-\int_t^{T_r} r_u du}\left(R(r_s,s,T_s)-K\right)|r_t\right]
$$

Applying Proposition 3, the value of this FRA is given by the following Proposition.

Proposition 8 *The value at time t of a FRA with \$1 notional value and maturity s, paying a fixed rate K and receiving a floating rate with tenor $T_s - s$, is given by*

$$\begin{aligned} FRA_t(r_t, s, T_r, T_s, K) &= P(r_t, t, T_r) E^s \left[R(r_s, s, T_s) - K | r_t \right] \\ &= P(r_t, t, T_r) \left[\frac{B(s, T_s)}{T_s - s} D(t, s) \left(1 + \frac{\xi}{\delta} \right) - \frac{\ln(A(s, T_s))}{T_s - s} - K \right] \end{aligned}$$

where

$$\xi = \frac{\delta B_s(t, s)}{D(t, s)} r_t$$

with $P(r_t, t, T_r)$ and $B(t, s)$ as given in Proposition 1 and $D(t, s)$ as given by (15). ■

2. **Interest rate swap and swaption**

An interest rate swap can be interpreted as either the difference between two coupon bonds or a portfolio of FRA's. Hence, swap valuation is a straightforward application of Proposition 1 or 8. Moreover, swaptions can be valued applying Proposition 6.

3. **Cap, floor, and collar**

A cap (floor) contract guarantees to its holder a pay-off if a certain floating interest rate is above (below) a specified rate, the cap (floor) level. Similarly to swaps, caps and floors involve a series of regular payments, usually referred as caplets or floorlets. Therefore, a cap (floor) can be interpreted as a portfolio of caplets (floorlets).

Consider a caplet written on a floating rate with \$1 face value and maturity s. If the caplet is exercised, the investor pays a fixed interest rate K and receives a floating rate

with tenor $T_s - s$. The floating rate is set at time s and the net cash-flow is received at time $T_r > s$.

Under the risk-neutral measure $\widetilde{P}$, the price at time t of this caplet is given by

$$Caplet_t(r_t, s, T_r, T_s, K) = \widetilde{E}\left[e^{-\int_t^{T_r} r_u du}\left(R(r_s, s, T_s) - K\right)^+ |r_t\right]$$

Under the s-forward measure P^s, the caplet price is given by the following Proposition.

Proposition 9 *The price at time t of a caplet written on the floating rate with \$1 face value and tenor $T_s - s$ is given as*

$$\begin{aligned} Caplet_t(r_t, s, T_r, T_s K) &= P(r_t, t, T_r) E^s\left[\left(R(r_s, s, T_s) - K\right)^+ |r_t\right] \\ &= P(r_t, t, T_r)\frac{D(t,s)B(s,T_s)}{\delta(T_s - s)}\left[\delta + \xi - 2e^{-\frac{\xi}{2}}\sum_{n=0}^{\infty}\left(\frac{\xi}{2}\right)^n \frac{\Gamma\left(\frac{\delta}{2} + n + 1, \frac{\rho}{2}\right)}{n!\,\Gamma\left(\frac{\delta}{2} + n\right)}\right] \\ &\quad - \left(\frac{\ln(A(s,T_s))}{T_s - s} + K\right) P(r_t, t, T_r)\left[1 - \chi^2(\rho, \delta, \xi)\right] \end{aligned}$$

where $\chi^2(\cdot)$ denotes the non-central chi-square distribution function, $\Gamma(\cdot)$ represents the Gamma function, and

$$\begin{aligned} \xi &= \frac{\delta B_s(t,s)}{D(t,s)} r_t \\ \rho &= \left(K + \frac{\ln(A(s,T_s))}{T_s - s}\right)\frac{\delta(T_s - s)}{B(s,T_s)D(t,s)} \end{aligned}$$

with $P(r_t, t, T_r)$ and $B(t,s)$ as given in Proposition 1 and $D(t,s)$ as given by (15). ■

In order to price a floorlet, same type of calculations as in this Proposition can be used. Alternatively, we can use the caplet-floorlet parity.

Cap, floor, and collar prices are a straightforward application of these results.

IV Conclusions

This paper has presented a new macro-financial continuous-time model for the term structure of interest rates assuming that the mean reversion level of interest rates and the interest rate volatility follow a cyclic behavior modeled by the harmonic waves functional form.

Our new model nests the original one presented in Cox it et al. (1985). In addition, keeping the same analytical tractability as in the CIR model, we provide more flexibility. In more detail, considering the relationship between interest rates and the business cycle, movements in the term structure of interest rates can be modeled in a more flexible way. For example, several humps can be easily achieved by choosing the appropriate parameters affecting the mean reversion level, that is, the amplitude of the wave, the temporal frequency, and the offset phase.

Consequently, we can value any contingent claim in a much more flexible framework while still maintaining the analytical tractability. Under these assumptions, we have computed closed-form expressions for prices of bonds and several fixed income and interest rate derivatives. We have also computed some risk management measures for bonds.

The results obtained have strong practical applications for pricing and risk management purposes and should be of special interest for traders, financial institutions and risk managers.

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Appendix of Proofs

Proof of Proposition 1

To solve equation (11), we guess an exponential-affine functional form for the bond price

$$P(r_t, t, T) = A(t, T)e^{-B(t,T)r_t}$$

with terminal conditions $A(T, T) = 1,\ B(T, T) = 0$. Then,

$$P_t = \frac{A_t}{A}P - B_t r_t P, \quad P_r = -BP, \quad P_{rr} = B^2 P$$

where arguments have been omitted and subscripts in functions P, A, and B denote partial derivatives. Replacing these expressions into (11), we get

$$\frac{A_t}{A} - B_t r_t - B\left[k(\theta_t - r_t) - \lambda_t r_t\right] + \frac{1}{2}\sigma_t^2 r_t B^2 - r_t = 0$$

with boundary condition $A(T, T) = 1,\ B(T, T) = 0$. Since this equation is linear in r_t, we obtain the following system of ordinary differential equations (ODEs)

$$B_t - (\lambda_t + k)\,B - \frac{1}{2}\sigma_t^2 B^2 + 1 = 0 \tag{23}$$

$$A_t - k\theta_t AB = 0 \tag{24}$$

Applying standard theory for Ricatti-type equations and defining $\tau = T - t$, the solution of (23) is given as $B(\tau) = \frac{v(\tau)}{u(\tau)}$ where $v(\tau)$ and $u(\tau)$ are solutions of the system

$$-v'(\tau) + u(\tau) - kv(\tau) = 0 \tag{25}$$

$$-u'(\tau) + \lambda_t u(\tau) + \frac{1}{2}\sigma_t^2 v(\tau) = 0 \tag{26}$$

Replacing the derivative of (25) into (26), we obtain the second-order ODE

$$v''(\tau) + bv'(\tau) + e(\tau)v(\tau) = 0 \tag{27}$$

where $b = k - \lambda_t$ and

$$e(\tau) = -\lambda_t k - \frac{1}{2} A_\sigma \sin^2(\varphi - wT + w\tau)$$

Setting $v(\tau) = g(\tau)M(\tau)$, expression (27) becomes

$$g(\tau)M''(\tau) + (2g'(\tau) + bg(\tau))M'(\tau) + (g''(\tau) + bg'(\tau) + e(\tau)g(\tau))M(\tau) = 0$$

that represents a Mathieu's differential equation if $2g'(\tau) + bg(\tau) = 0$. Then, we get

$$g(\tau) = ce^{\frac{1}{2}(\lambda - k)\tau}$$

with arbitrary constant c. Then, we obtain

$$v(\tau) = e^{\frac{1}{2}(\lambda - k)\tau} \left(c_1 MC(a, q, x) + c_2 MS(a, q, x)\right) \tag{28}$$

where MC and MS represent the Mathieu cosine and sine functions, respectively, and

$$a = -\frac{A_\sigma + (\lambda + k)^2}{4w^2},\ q = -\frac{A_\sigma}{8w^2},\ x = \varphi - wT + w\tau$$

The boundary condition $B(0) = 0$ implies $v(0) = 0$. Then, choosing $c_2 = 1$ in (28) implies

$$c_1 = -\frac{MS(a, q, \varphi - wT)}{MC(a, q, \varphi - wT)}$$

Substituting (28) and its derivative into (25), we get

$$u(\tau) = e^{\frac{1}{2}(\lambda - k)\tau} \left[\frac{1}{2}(\lambda + k)\left(c_1 MC(a, q, x) + MS(a, q, x)\right) + w\left(c_1 MCP(a, q, x) + MSP(a, q, x)\right)\right] \tag{29}$$

where MCP and MSP represent the derivatives with respect to x of the Mathieu cosine and sine functions, respectively. Therefore, using expressions (28)-(29),we get $B(\tau)$.

Finally, equation (24) immediately provides $A(t, T) = \exp\left\{\int_t^T k\theta_t B(t, T)dt\right\}$. ■

Proof of Proposition 5

From Proposition 3, we know

$$c_t(r_t, s, T, K) = P(r_t, t, s)E^s\left[\left(A(s,T)e^{-\frac{1}{\delta}B(s,T)D(t,s)\varpi} - K\right)^+ |r_t\right]$$

where E^s represents expectation with respect to the s-forward measure P^s. Hence,

$$\begin{aligned} c_t(r_t, s, T, K) &= P(r_t, t, s)\int_0^\infty \left(A(s,T)e^{-\frac{1}{\delta}B(s,T)D(t,s)\varpi} - K\right)^+ d\chi^2(\varpi, \delta, \xi_1) \\ &= P(r_t, t, s)\int_0^{\rho_1} A(s,T)e^{-\frac{1}{\delta}B(s,T)D(t,s)\varpi} d\chi^2(\varpi, \delta, \xi_1) - KP(r_t, t, s)\chi^2(\rho_1, \delta, \xi_1) \end{aligned}$$

where $\chi^2(\cdot)$ denotes the non-central chi-square distribution function and

$$\begin{aligned} \xi_1 &= \frac{\delta B_s(t,s)}{D(t,s)} r_t \\ \rho_1 &= \frac{\delta}{B(s,T)D(t,s)} \ln\left(\frac{A(s,T)}{K}\right) \end{aligned}$$

Using the expression for the density function of a non-central chi-square distribution, the integral becomes

$$P(r_t, t, s)A(s,T)\int_0^{\rho_1} e^{-\frac{1}{\delta}B(s,T)D(t,s)\varpi}\left[2^{-\frac{\delta}{2}}\varpi^{\frac{\delta}{2}-1}e^{-(\varpi+\xi_1)/2}\sum_{n=0}^{\infty}\frac{\varpi^n(\xi_1)^n}{n!4^n\Gamma(\frac{\delta}{2}+n)}\right]d\varpi$$

Considering the change of variable $y = \left(\frac{2}{\delta}B(s,T)D(t,s)+1\right)\varpi$ and defining

$$\begin{aligned} \rho_2 &= \rho_1\left(\frac{2}{\delta}B(s,T)D(t,s)+1\right) \\ \xi_2 &= \frac{\rho_1}{\rho_2}\xi_1 \end{aligned}$$

we get

$$P(r_t, t, s)F(r_t, t, s, T)\chi^2(\rho_2, \delta, \xi_2)$$

with $F(r_t, t, s, T)$ as given by Proposition 4. ∎

Proof of Proposition 9

From Proposition 3, we know

$$
\begin{aligned}
& Caplet_t(r_t, s, T_r, T_s, K) = P(r_t, t, T_r) E^s \left[(R(r_s, s, T_s) - K)^+ \,|r_t \right] \\
= \; & P(r_t, t, T_r) \int_\rho^\infty \frac{B(s, T_s) D(t, s)}{\delta(T_s - s)} \varpi d\chi^2(\varpi, \delta, \xi) - \left(\frac{\ln(A(s, T_s))}{T_s - s} + K \right) P(r_t, t, T_r) \left[1 - \chi^2(\rho, \delta, \xi) \right]
\end{aligned}
$$

where E^s represents expectation with respect to the s-forward measure P^s, $\chi^2(\cdot)$ denotes the non-central chi-square distribution function, and

$$
\begin{aligned}
\xi &= \frac{\delta B_s(t, s)}{D(t, s)} r_t \\
\rho &= \left(K + \frac{\ln(A(s, T_s))}{T_s - s} \right) \frac{\delta(T_s - s)}{B(s, T_s) D(t, s)}
\end{aligned}
$$

Using the expression for the density function of a non-central chi-square distribution, the integral becomes

$$
\frac{B(s, T_s) D(t, s)}{\delta(T_s - s)} 2^{-\frac{\delta}{2}} e^{-\frac{\xi}{2}} \sum_{n=0}^{\infty} \frac{\xi^n}{n! 4^n \Gamma(\frac{\delta}{2} + n)} \int_\rho^\infty \varpi^{\frac{\delta}{2} + n} e^{-\frac{1}{2}\varpi} d\varpi \tag{30}
$$

Note that

$$
\int_\rho^\infty \varpi^{\frac{\delta}{2} + n} e^{-\frac{1}{2}\varpi} d\varpi = 2^{\frac{\delta}{2} + n + 1} \left[\Gamma\left(\frac{\delta}{2} + n + 1 \right) - \Gamma\left(\frac{\delta}{2} + n + 1, \frac{\rho}{2} \right) \right]
$$

where $\Gamma(\alpha, \varrho) = \int_0^\varrho e^{-t} t^{\alpha - 1} dt$ and $\Gamma(\alpha) \equiv \Gamma(\alpha, \infty)$. Using the Taylor expansion of the exponential function and straightforward algebra, (30) becomes

$$
\frac{B(s, T_s) D(t, s)}{\delta(T_s - s)} \left[\delta + \xi - 2 e^{-\frac{\xi}{2}} \sum_{n=0}^{\infty} \left(\frac{\xi}{2} \right)^n \frac{\Gamma\left(\frac{\delta}{2} + n + 1, \frac{\rho}{2} \right)}{n! \; \Gamma\left(\frac{\delta}{2} + n \right)} \right]
$$

■

Appendix of Figures

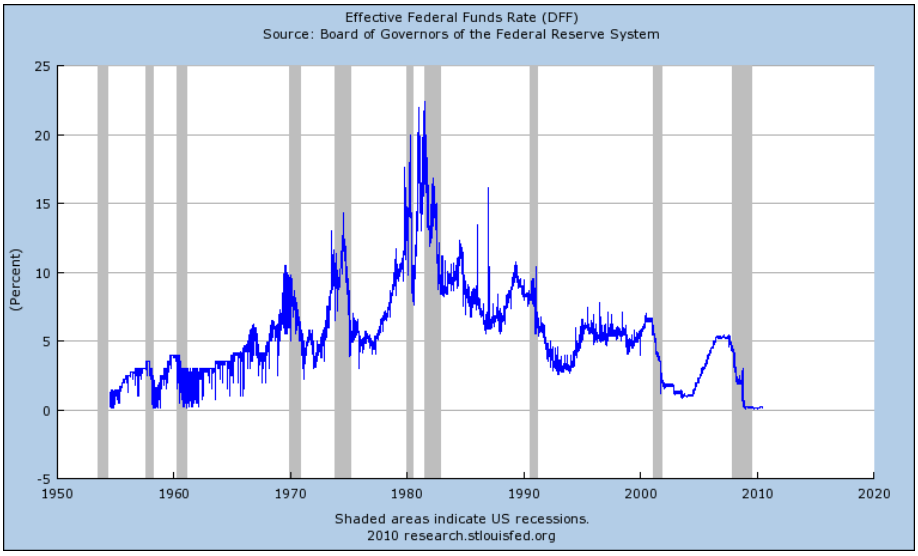

Figure 1: Evolution in time of the effective Federal funds rate up to 2010.

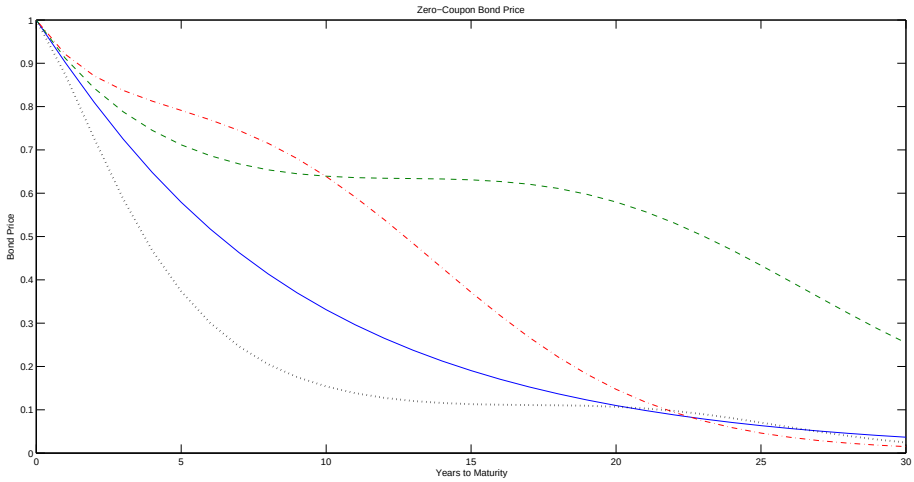

Figure 2: Zero-coupon bond price. The parameters for the CIR model (solid line) are $r_0 = 0.1$, $k = 0.15$, $\theta = 0.15$, $\sigma = 0.15$, $\lambda = 0$. Additionally, we consider three alternatives in our model: a) $k = 0.6$, $A_\sigma = 0.0225$, $A_\theta = 0.15$, $\varphi = \pi/4$, $w = 2\pi/90$ (dashed line), b) $k = 0.4$, $A_\sigma = 0.02$, $A_\theta = 0.25$, $\varphi = \pi$, $w = 2\pi/90$ (dashed-dotted line), c) $k = 0.4$, $A_\sigma = 0.03$, $A_\theta = 0.3$, $\varphi = \pi/2$, $w = 2\pi/60$ (dotted line).

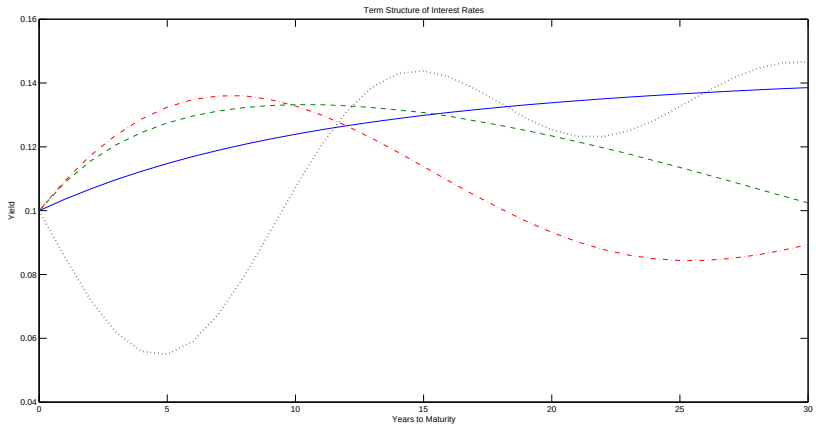

Figure 3: Term Structure of Interest Rates. CIR parameters (solid line): $r_0 = 0.1,\ k = 0.15,\ \theta = 0.15,\ \sigma = 0.015,\ \lambda = 0$. Three alternatives in our model: a) $k = 0.4,\ A_\sigma = 0.002,\ A_\theta = 0.15,\ \varphi = \pi/2,\ w = 2\pi/160$ (dashed line), b) $k = 0.2,\ A_\sigma = 0.002,\ A_\theta = 0.2,\ \varphi = \pi/2,\ w = 2\pi/60$ (dashed-dotted line), c) $k = 0.4,\ A_\sigma = 0.002,\ A_\theta = 0.3,\ \varphi = \pi/10,\ w = 2\pi/30$ (dotted line).

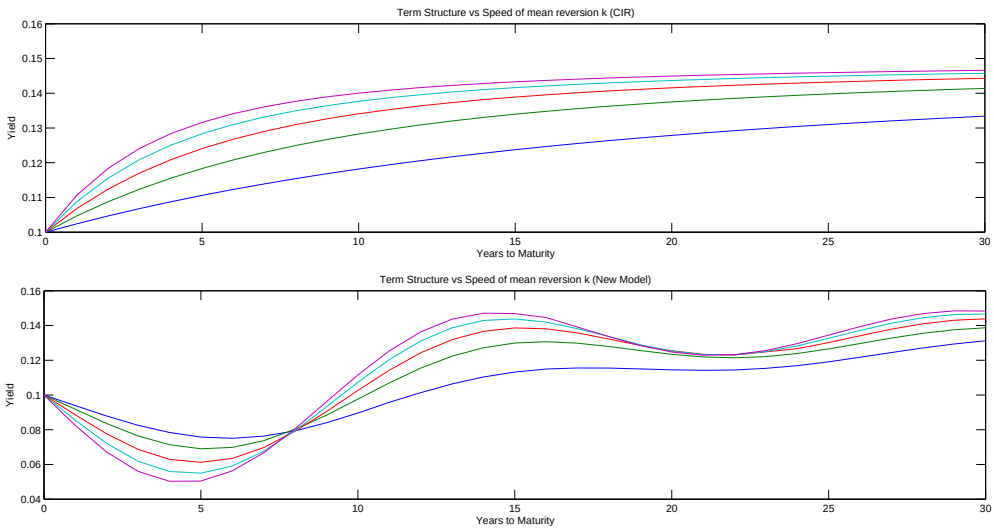

Figure 4: Term structure for different values of k. CIR parameters: $r_0 = 0.1,\ \theta = 0.15,\ \sigma = 0.015,\ \lambda = 0$. Values of k (curves from the top down) are $0.5, 0.4, 0.3, 0.2,$ and 0.1, respectively. Parameters for the new model: $r_0 = 0.1,\ A_\sigma = 0.002,\ A_\theta = 0.3,\ \varphi = \pi/10,\ w = 2\pi/30,\ \lambda = 0$. The values of k corresponding to different curves are $0.5, 0.4, 0.3, 0.2,$ and 0.1.

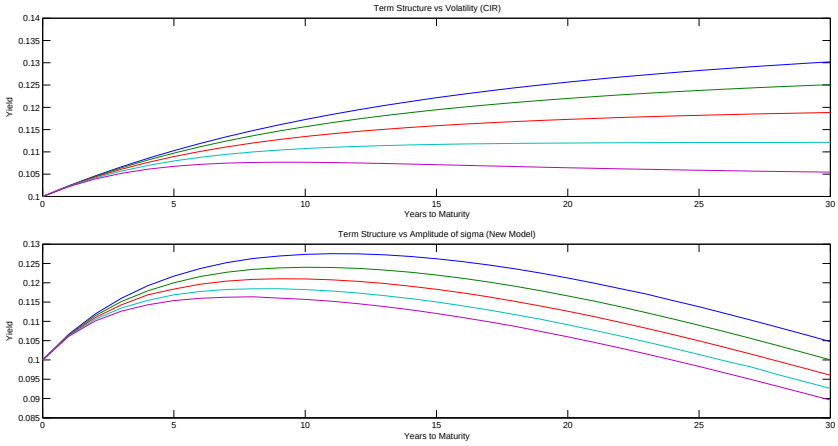

Figure 5: Term structure for different values of σ. CIR parameters: $r_0 = 0.1$, $\theta = 0.15$, $k = 0.1$, $\lambda = 0$. Values of σ (curves from the top down): $0.035, 0.055, 0.075, 0.095$, and 0.115, respectively. New model parameters: $r_0 = 0.1$, $k = 0.3$, $A_\theta = 0.15$, $\varphi = \pi/2$, $w = 2\pi/180$, $\lambda = 0$. Values of A_σ (curves from the top down): $0.011, 0.021, 0.031, 0.041$, and 0.051, respectively.

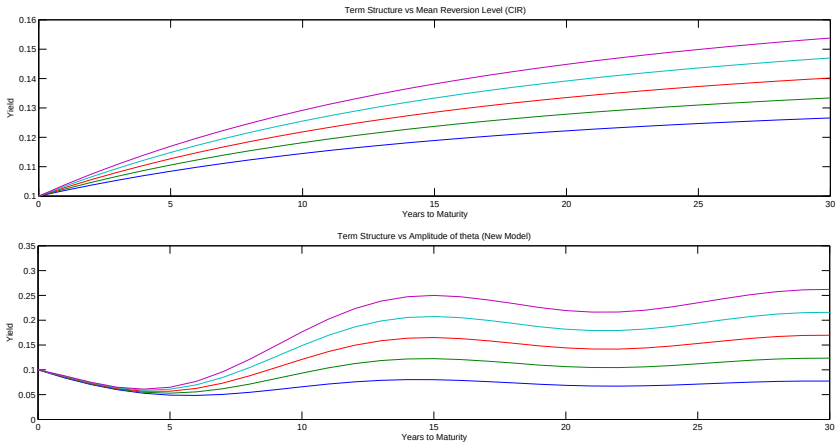

Figure 6: Term structure for different values of θ in the CIR model and different values of A_θ in our model. CIR parameters: $r_0 = 0.1$, $\sigma = 0.015$, $k = 0.10$, $\lambda = 0$. Values of θ (curves from the top down): $0.18, 0.17, 0.16, 0.15$, and 0.14, respectively. Parameters for the new model: $r_0 = 0.1$, $k = 0.4$, $A_\sigma = 0.002$, $\varphi = \pi/10$, $w = 2\pi/30$, $\lambda = 0$. Values of A_θ (curves from the top down): $0.55, 0.45, 0.35, 0.25$, and 0.15, respectively.

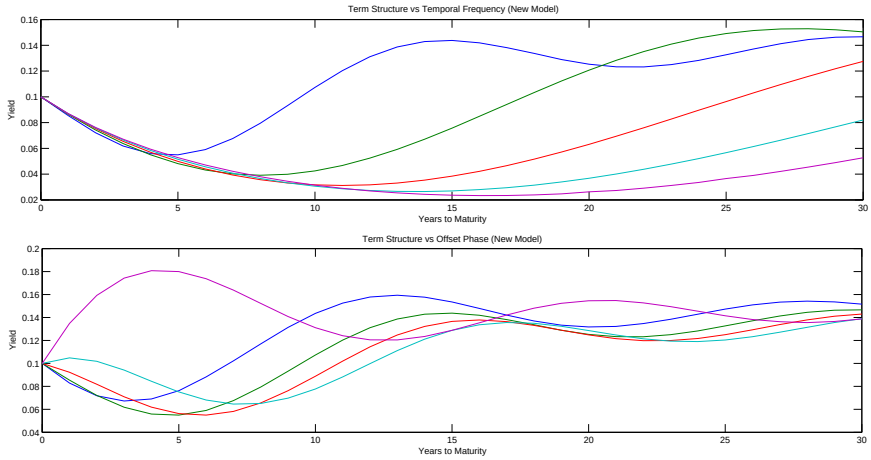

Figure 7: Term structure for different values of w and φ in our model. In both graphs, $r_0 = 0.1, A_\theta = 0.3, k = 0.4, A_\sigma = 0.002, \lambda = 0$. In the first graph, $\varphi = \pi/10, w = 2\pi/30, 2\pi/60, 2\pi/90, 2\pi/120, 2\pi/150$. In the second graph, $w = 2\pi/30, \varphi = \pi, \pi/10, \pi/6, \pi/4, \pi/2$.