DYNAMIC EQUITY ALLOCATION FOR INSURANCE COMPANIES IN THE PRESENCE OF SOLVENCY II CONSTRAINTS

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Abstract

This paper introduces a framework for designing dynamic asset allocation strategies that can be used by insurance companies within the Solvency II regulatory framework to achieve a substantial exposure to equity risk, and the associated premium, while taking into account the presence of stochastically time-varying risk and return parameters. Extensive numerical and empirical simulations confirm that the opportunity cost of static portfolio strategies ignoring return predictability is substantial in the presence of Solvency II constraints. Our results have important potential implications for the design of improved forms of dynamic investing solutions that could help insurance companies facing exceedingly high capital charges associated with equity investments.

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1. Introduction

The Solvency II directive of the European Parliament and Council (2009) introduces a prudential framework for the computation of regulatory capital requirements of insurers within the European Union. The objective of the new Solvency II regulatory framework, which should come into force at the beginning of 2013, is to introduce unified economic risk-based solvency requirements for insurance companies across all EU member states.

As a risk-based methodology, the Solvency II regulatory framework defines the solvency impact of investing in various asset classes. These solvency conditions are defined over a short-term horizon in that they aim at ensuring that the insurance company has enough free capital to cover a loss that could occur due to an instantaneous extreme event over the following year. More specifically, the directive sets the risk-based regulation for European insurance companies. Capital requirements are calculated, such as the company's assets remaining above its liabilities with a 99.5% probability over a one-year horizon (which is summarized as a 99.5% 1-year VaR requirement). Insurers will be able to either use the "standard formula" to apply all relevant stress tests that define the extreme market event, or model the overall balance sheet and look at the 99.5% Value-at-Risk over one year of the surplus, formally defined as the difference between asset and liability values. For each identified risk factor, the extreme scenario is calibrated as being the worst loss that could occur over 1 year.

The standard formula of the Solvency II framework makes investment in equity very costly, since the capital charge evaluated as the 99.5% VaR is taken to be a 39% loss as a base case. In practice, the estimation of capital charges for equities takes into account the so-called dampener effect, see article 106 of the directive of the European Parliament and Council (2009), and varies in a counter-cyclical manner in the 29%-49% range. Life insurance companies that have ring-fenced retirement provisions can also reduce capital requirements via the duration-based approach, which allows, subject to prior supervisory approval, the risk-charge to be calculated with a long-term equity volatility assumption that is lower than that in the standard formula. Despite these two opportunities for decreasing the regulatory cost of capital, capital charges for equity investment within Solvency II remain extremely high, and are often described as "prohibitive" by insurance companies.

In reaction to the forthcoming implementation of this prudential framework, some large insurance companies have already started to reduce their exposure to equity risk so as to prepare for the new regulatory constraints. While this forced shift away from access to the equity risk premium implies significant opportunity costs for insurance companies and their clients, it is beyond the scope of the present paper to question the validity and relevance of imposing a particularly strong penalty on equity investments.[2] The focus of this paper is instead to take these regulatory constraints as given, and analyze, in a stylized partial equilibrium continuous-time setting, how dynamic allocation strategies can be designed to allow insurance companies to maintain a non-trivial equity exposure without an exceedingly high associated cost in terms of capital consumption from the regulatory perspective.

Our approach to long-term investing in the presence of Solvency II constraints builds upon an abundant stream of research on long-term investment decisions in the presence of a stochastic opportunity set. The development by Merton(1969,1971) of dynamic portfolio theory in the late

[2] See for example, Dumas et al. (2009) for a general equilibrium framework that can be used to analyze the impact of financial regulations.

sixties and early seventies, following pioneering work by Hakansson(1969,1971) and Samuelson (1969), has led to a number of fundamental new insights with respect to simple static portfolio selection models (see Markowitz (1952)). In early seminal work, Merton (1971) has shown that the presence of a stochastic opportunity set leads non-myopic long-term investors to incorporate intertemporal hedging demands in addition to the standard speculative motive. Subsequent papers have shown that when maximizing utility from terminal (nominal) wealth, an investor only hedges those state variables that impact the nominal short-term rate and the market prices of risk (see for example Detemple et al. (2003)). Some papers, e.g. Lioui and Poncet (2001), Munk and Sørensen (2004), have explicitly solved the portfolio choice problem when only one of the two state variables is stochastic, with a separate analysis of the impact of interest rate risk. Other papers have solved the impact of a mean-reverting equity risk premium - see Kim and Omberg (1996) for the incomplete market case with utility from terminal wealth only, and Wachter (2002) for the complete market case with utility from intermediate consumption. More realistic continuous-time models have subsequently been introduced to simultaneously account for the presence of uncertainty on both interest rates and market prices of risk, either by solving numerically the Hamilton-Jacobi-Bellman equation obtained through dynamic programming - see Brennan et al. (1997), or more recently, by exploiting the affine structure of the model in the sense of Dai and Singleton (2000) and Liu (2007) and solving the HJB equation explicitly, e.g. Munk et al. (2004) and Sangvinatsos and Wachter (2005).

One of the main qualitative findings that perhaps stands out from the literature on long-term portfolio decisions with return predictability is the fact that equities serve as a hedge against unfavorable equity returns in the presence of mean-reverting returns. As a result, the optimal allocation to stocks is higher compared to the myopic case, and investors with longer time-horizon hold more stocks compared to investors with shorter horizon - see Kim and Omberg (1996) and Wachter (2002). One key problem, however, is that this prescription can lead to extremely difficult situations when risk is assessed from a shorter-term perspective, in particular in the context of a severe bear equity market such as the one experienced in 2008. In fact, it appears that most, if not all, investors even those with the longest possible horizons (such as insurance companies, but also pension funds) inevitably face a number of regulatory (or otherwise) short-term performance constraints. The Solvency II regulation is a clear example of such constraints, as it imposes strict risk budgets over a one-year horizon.

In other words, it appears that one key element missing from the literature on life-cycle investing is how the presence of short-term performance constraints faced by investors would affect the optimal long-term allocation decisions. Fortunately, the problem of portfolio selection with performance constraints has in fact been examined by a totally separate strand of the literature, which has shown that the presence of such constraints leads long-term investors to adjust the allocation to risky versus risk-free assets in a state-dependent manner as a function of a suitably-defined risk budget state variable. The literature on portfolio choice with minimum performance constraints starts with the literature on portfolio insurance, which focuses on strategies that aims to guarantee a given minimum level of terminal wealth, based either on Constant-Proportion Portfolio Insurance (CPPI) introduced by Black and Jones (1987) and Black and Perold (1992), or on Option-Based Portfolio Insurance (OBPI) strategies, see Leland (1980). More recently, Grossman and Zhou (1996) show that CPPI strategies are optimal for investors who maximize their expected utility from terminal wealth subject to the constraint that this wealth should be greater than or equal to some fixed floor. In a related effort, Basak (2002) suggests introducing an implicit rather than an explicit constraint on terminal wealth, which amounts to the assumption that marginal utility goes smoothly to infinity as

wealth shrinks to the floor. Several papers have generalized these models by imposing minimum performance constraints that involves choosing a stochastic, as opposed to a deterministic, benchmark. Teplá (2001) shows that the optimal strategy in the presence of such constraints involves a long position in an exchange option. Basak et al. (2006) generalize the approach by considering less stringent constraints on performance, i.e. by imposing that the minimum performance should be achieved with a given probability, not necessarily with probability 1. In all these studies, however, investment opportunities are constant so that optimal strategies in the absence of performance constraints are of the fixed-mix nature, and the presence of return predictability is simply not accounted for.[3]

In what follows, we mix these various ingredients and analyze a long-horizon dynamic allocation model in the presence of short-term Solvency II constraints, as well as stochastic interest rates, mean-reverting equity risk premium and volatility. Because solving the optimal allocation problem in the presence of both a stochastic opportunity set and short-term constraints may not be feasible in analytical form, we use a different approach. More specifically, we first use the martingale approach to portfolio optimization in incomplete market settings to solve the problem in the absence of short-term constraints. In this context, we extend Munk et al. (2004) by relaxing the simplifying assumption of a perfect negative correlation between equity returns and risk premium uncertainty - see for example, Sangvinatsos and Wachter (2005). While our setting is rich enough to account for the aforementioned features, we manage at this stage to maintain a (quasi) explicit representation for the optimal portfolio strategy. In a second step, we introduce a stylized risk budget constraint directly drawn from an analysis of the Solvency II regulation, and introduce a simple transformation of the optimal unconstrained strategy, directly inspired by the form of the optimal portfolio strategy with performance constraints in the case of constant opportunity set, that would lead to a strategy respecting the Solvency II constraints.

While the resulting strategy, which we call “constrained” strategy, cannot to be regarded as optimal in a formal sense, it enjoys a number of desirable properties; in particular it allows for a potentially substantial allocation to equities while respecting a given capital charge risk budget. Besides, one of the key benefits of this approach is that it allows us to maintain quasi-analytical expressions for the strategy in the presence of short-term constraints. Having access to analytical solutions turns out to be critically useful for any attempt to perform Monte-Carlo simulations, which is one of the formal requirements by the regulation for insurance companies that want to use internal models and deviate from the standard formula.

From a qualitative standpoint, our main finding is to show that long-term objectives and short-term constraints need to be mutually exclusive. In fact, our analysis shows that the rational response to the Solvency II regulation should not be a dramatic static decrease of the allocation to equity but instead a time-varying allocation to equity that will be both a function of the current opportunity set, of the time-horizon, as well as of the outstanding risk budget at the current time.

[3] An extension to the case of stochastic opportunity set has been proposed by El Karoui et al. (2005), who show that OBPI maximizes expected utility in a more general framework where risk and return parameters are not necessarily constant. However, they adopt a reduced-form approach, directly modeling the unconstrained strategy. In particular, the impact of stock return predictability on the optimal unconstrained strategy does not appear, while it is at the heart of the conflict between long-term objectives and short-term constraints.

The results of an analysis based on 10,000 Monte Carlo simulations show that the average returns achieved by the Solvency II allocation strategies are increasing in the capital charge, which was expected since the average stock allocation increases also in the Solvency II risk budget. The wealth allocated to equities, and therefore the average performance, is also an increasing function of the time-to-horizon, which can be explained by the decreasing term-structure of equity risk implied by the presence of mean-reversion in equity returns. Finally, even though the dynamic portfolio strategy is implemented in a discrete time (monthly), we find that there is no violation of the target Solvency II risk budgets at the 99.5% confidence level, and in fact no violation at the 100% level given our scenarios. In that sense, the Solvency II benchmarks achieve the initial objective, that is allowing for a substantial allocation to equities while respecting given Solvency II risk budgets.

In order to have a better understanding of the opportunity costs involved in following standard static asset allocation strategies, as opposed to using dedicated dynamic asset allocation benchmarks that have been specifically engineered to allow for the optimal spending of the regulatory risk budgets, we turn to the dual analysis. We consider the static benchmark that has an equity allocation leading to a given level of Solvency II capital requirement (using the standard formula of 39% for equity), and then look at the corresponding average returns. The results obtained for such strategies show that the static allocations have a performance level substantially lower than that of the comparable Solvency II benchmarks. Moreover, we see that the 99.5% max losses computed from our 10,000 Monte Carlo simulations are always higher than the capital requirement obtained from the Solvency II standard formula, which suggests that lower stock allocations should be used, leading to even lower performances for the static benchmarks. We finally confirm the robustness of these results by analyzing the performance of the strategies over historical datasets.

The paper is organized as follows. In section 2, we introduce the formal model. In section 3, we present the methodology we use to calibrate the model and the results we obtain. In section 4, we perform numerical and empirical testing of the model, while some concluding thoughts can be found in section 5.

2. A Formal Dynamic Allocation Model in the Presence of Solvency II Constraints

In this section, we solve the long-term optimization program of an institutional investor, interpreted here as an insurance company, which faces stochastic investment opportunities. We first introduce the setting, and then present the optimal strategy in case of market incompleteness coming from possibly imperfect correlation between stock returns and stock risk premium, volatility risk, and we finally introduce Solvency II-type constraints.

2.1 State variables and asset returns

We consider an investor with a finite horizon T. For simplicity, and also because the focus of the paper is on equity investments, we assume that the investors has access to two asset only, the risk-free asset and a stock index. The nominal short-term interest rate is assumed to follow a mean-reverting process, of the type proposed by Vasicek (1977):

$$\mathrm{d}r_t = a(b - r_t)\,\mathrm{d}t + \sigma^r\,\mathrm{d}z_t^r \tag{2.1}$$

The locally risk-free asset, which we shall refer to as cash, has a value given by:

$$S_t^0 = \exp\left[\int_0^t r_s \mathrm{d}s\right]$$

Since it has zero duration, the cash does not provide any hedge against interest rate risk. In order to have a hedging instrument against interest rate risk, one would have to use nominal bonds with positive duration. In fact, because the Vasicek model considered here has only one factor, a single constant-maturity bond would be sufficient to replicate the entire term structure. While other papers assume that a zero-coupon is traded, see Sørensen (1999) and Munk et al. (2004), we will not make that assumption here for the aforementioned reasons. The dynamics of the stock price index S is given by:

$$\frac{\mathrm{d}S_t}{S_t} = \left(r_t + \sigma_t^S \lambda_t^S\right)\mathrm{d}t + \sigma_t^S\,\mathrm{d}z_t^S \tag{2.2}$$

As in Kim and Omberg (1996), Wachter (2002) and Munk et al. (2004), we assume that the conditional Sharpe ratio of that stock follows a mean-reverting process:[4]

$$\mathrm{d}\lambda_t^S = \kappa\left(\overline{\lambda} - \lambda_t^S\right)\mathrm{d}t + \sigma^\lambda\,\mathrm{d}z_t^\lambda \tag{2.3}$$

If one assumes, as in Wachter (2002), that innovations to this state variable are perfectly anti-correlated with the innovations to stock returns, i.e. if $\rho^{S\lambda} = -1$, then Sharpe ratio risk is spanned. In the remainder of section 2, we do not assume such a perfectly negative correlation.

Our model also allows the conditional variance of stock returns to be stochastic. This feature was not present in the related literature except in Kim and Omberg (1996), Liu (2007) and Chacko and Viceira (2005), but we deem it to be an important component of a long-term investment model, given the large empirical evidence for time-variation in the volatility of stock returns. We follow Heston (1993) by assuming that the conditional variance $V = \left(\sigma^S\right)^2$ of stock returns follows a mean-reverting square-root process:

$$\mathrm{d}V_t = \alpha\left(\overline{V} - V_t\right)\mathrm{d}t + \sigma^V\sqrt{V_t}\,\mathrm{d}z_t^V \tag{2.4}$$

The square root in the diffusion term prevents the process from going negative.

The equations that describe the model can also be written in terms of a four-dimensional Brownian motion $\underline{z}$ and of the *unit* volatility vectors $\underline{\rho}^i$, that are defined by:[5]

$$\mathrm{d}z_t^i = \left(\underline{\rho}^i\right)'\mathrm{d}\underline{z}_t, \quad i = \{r, S, \lambda, V\}$$

A market price of risk vector is obtained as:

[4]Munk et al. (2004) assume in fact that the expected excess return on the stock, not its Sharpe ratio, is a mean-reverting process. However, since the volatility of the stock is constant in their model, it is equivalent to assume mean-reversion in the expected excess return or in the Sharpe ratio.

[5]We warn the reader that we are using the unit volatility vectors (whose norm is by definition equal to 1), rather than the standard volatility vectors (whose norm is equal to the volatility of the process).

$$\underline{\lambda}_t = \lambda_t^S \underline{\rho}^S$$

As shown by He and Pearson (1991), the other market prices of risk are those vector processes of the form $\underline{\lambda} + \underline{\nu}$, where $\underline{\nu}$ satisfies $\left(\underline{\rho}^S\right)'\underline{\nu}_t = 0$ almost surely for all date t. Each of these prices of risk defines a pricing kernel M, through:

$$M_t = \exp\left[-\int_0^t \left(r_s + \frac{\left\|\underline{\lambda}_s + \underline{\nu}_s\right\|^2}{2}\right)\mathrm{d}s - \int_0^t \left(\underline{\lambda}_s + \underline{\nu}_s\right)' \mathrm{d}\underline{z}_s\right], \quad t \le T$$

There are infinitely many pricing kernels, since there are infinitely many vectors orthogonal to $\underline{\rho}^S$.

2.2 Derivation of the optimal unconstrained strategy

We let w_t denote the weight allocated to the locally risky asset, which is the stock index. The weight allocated to the cash is thus $1 - w_t$. With these notations, we can write the budget constraint of an investor who receives no non-financial income and does not consume as:

$$\mathrm{d}A_t = A_t\left[\left(r_t + w_t\sigma_t^S\lambda_t^S\right)\mathrm{d}t + w_t\sigma_t^S\mathrm{d}z_t^S\right] \tag{2.5}$$

Throughout the paper, we let $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ be the Constant Relative Risk Aversion (CRRA) utility function. We assume that the objective of the insurance company is assumed to focus on the maximization of the risk/return trade-off in terminal wealth, while risk constraints with respect to the liabilities will be handled separately, as explained in the next sub-section:

$$\max_w E\left[U\left(A_T\right)\right] \tag{2.6}$$

subject to budget constraint (2.5). It should be emphasized at this stage that our ambition is not to develop a full-fledged asset-liability management model for insurance companies, which would be extremely challenging given the complexity of insurance companies' liabilities. Instead our objective is to develop a stylized asset-only model for equity allocation by an insurance company facing Solvency II constraints.

(2.6) is a dynamic problem since the control variable is the process of weights allocated to the stock index. In the martingale approach initially developed by Cox and Huang (1989), the dynamic program is replaced by an equivalent static program where the control variable is the terminal wealth and the budget constraint is expressed in terms of the present value of the terminal wealth. But the market is incomplete, so there are infinitely many pricing kernels in the economy, hence infinitely many static problems. As shown by He and Pearson (1991), the original dynamic problem (2.6) is equivalent to the static problem obtained with one particular pricing kernel M^*, known as the "minimax" pricing kernel. Formally, the portfolio choice problem is equivalent to:

$$\max_{A_T} E\left[U\left(A_T\right)\right], \quad \text{subject to } E\left[M_T^* A_T\right] = A_0$$

and the optimal portfolio strategy is the one that replicates the optimal payoff. The following proposition gives a full description of this optimal strategy.

Proposition 1. Consider the optimization problem (2.6) subject to budget condition (2.5).

- The optimal payoff is given by:

$$A_T^* = \frac{A_0}{E\left[\left(M_T^*\right)^{1-\frac{1}{\gamma}}\right]}\left(M_T^*\right)^{-\frac{1}{\gamma}} \tag{2.7}$$

where M^* is the minimax pricing kernel, and the optimal wealth process reads:

$$A_t^* = \frac{A_0}{E\left[\left(M_T^*\right)^{1-\frac{1}{\gamma}}\right]}\left(M_t^*\right)^{-\frac{1}{\gamma}} g\left(t, r_t, \lambda_t^S\right) \tag{2.8}$$

where g is a function of time and the equity Sharpe ratio given by:

$$g\left(t,r,\lambda\right) = \exp\left\{\frac{1-\gamma}{\gamma}\left[C_1(T-t;\gamma) + C_2(T-t;\gamma)r + C_3(T-t;\gamma)\lambda + \frac{1}{2}C_4(T-t;\gamma)\lambda^2\right]\right\}$$

- The optimal allocation to stock is given by:

$$w_t^* = \frac{\lambda_t^S}{\gamma\sigma_t^S} - \left(1-\frac{1}{\gamma}\right)C_2\left(T-t\right)\frac{\rho^{Sr}\sigma^r}{\sigma_t^S} - \left(1-\frac{1}{\gamma}\right)\left[C_3(T-t;\gamma) + C_4(T-t;\gamma)\lambda_t^S\right]\frac{\rho^{S\lambda}\sigma^\lambda}{\sigma_t^S} \tag{2.9}$$

with the remainder of wealth invested in cash.

- The functions $C_1(\cdot;\gamma)$, $C_2(\cdot;\gamma)$, $C_3(\cdot;\gamma)$ and $C_4(\cdot;\gamma)$ are the solutions to a system of (coupled) ordinary differential equations (ODEs) given in 0

Proof. See Appendix

We now discuss the structure of the optimal allocation in more detail. The first term $\frac{\lambda_t^S}{\gamma\sigma_t^S}$ on the right-hand side of equation (2.9) is already present when investment opportunities are constant (see Merton (1969). It represents the speculative demand for stocks. The only difference from the constant opportunity economy is that the weight allocated to stocks in this term has become a function of the current Sharpe ratio and volatility. The speculative demand increases when the risk-return ratio of the stock is more attractive, and it is decreasing in the volatility. An investor with logarithmic utility (which is obtained for $\gamma = 1$) would set the weight allocated to the stock equal to the speculative demand.

The second term $-\left(1-\frac{1}{\gamma}\right)C_2\left(T-t\right)\frac{\rho^{Sr}\sigma^r}{\sigma_t^S}$ on the right-hand side of equation (2.9) can be regarded as the hedging demand against interest rate risk of insurance companies that have an investment horizon of *T* years. It is clear from the separation formula that for $\gamma > 1$, the hedging demand against

interest rate risk is increasing in the relative risk aversion γ, which makes sense: more risk-averse investors want to hold more of the portfolio that is risk-free over the long run. In general, this hedging demand rationalizes the investment in a pure discount bond with a maturity date matching the investor's horizon.[6] In the context of this analysis, however, we focus on the equity allocation for insurance companies, and recognize that insurance companies manage interest rate risk separately, in the context of their fixed-income portfolio. In the absence of fixed-income securities, some hedging demand can still exist for an equity-only portfolio if equity returns have a non-zero correlation with interest rates. On the other hand, if the correlation ρ^{Sr} is zero, then the hedging demand cancels out, as can be seen from the fact that this hedging demand is proportional to that term. In the empirical calibration exercise, we actually take the correlation between stock returns and interest rates to be zero so that the optimal allocation strategy that we consider in what follows does not contain a hedging demand against interest rate risk.

The last term $\left(1-\frac{1}{\gamma}\right)\left[C_3(T-t;\gamma)+C_4(T-t;\gamma)\lambda_t^S\right]\frac{\rho^{S\lambda}\sigma^{\lambda}}{\sigma_t^S}$ on the right-hand side of equation (2.9) involves the quantity $\beta_t^{\lambda} \equiv \frac{\rho^{S\lambda}\sigma^{\lambda}}{\sigma_t^S}$, which is the beta of the stochastic Sharpe ratio with respect to the stock index. Hence this term represents a hedging demand against Sharpe ratio risk. With reasonable parameter values, the correlation $\rho^{S\lambda}$ will be negative and the quantities $C_3(T-t;\gamma)$ and $C_4(T-t;\gamma)$ will be positive.[7] Hence the term represents an excess demand for stocks that arise from the uncertainty in the value of future Sharpe ratios. From the ODEs given in 0, we see that these quantities depend on both the interest rate and Sharpe ratio models. Hence the non-logarithmic investor ($\gamma \neq 1$) must not only know the current opportunity set, i.e. the risk and return characteristics of the various assets over the next trading interval, but also how these characteristics evolve over time.

Overall, the decomposition formula of Proposition 1 is similar to the one introduced by Detemple and Rindisbacher (2010). The first term is the speculative demand for the locally risky asset. The second term represents an investment in the portfolio that best replicates the asset that is safe over the long-run, here a zero-coupon bond of maturity T. The third term is a hedging demand against the uncertainty in prices of risk.

It is clear from Proposition 1 that the weight allocated to the stock is not only time-dependent, but also state-dependent, in contrast to the heuristic deterministic glide paths of existing life-cycle funds. In particular, the optimal allocation is a function of the stock index Sharpe ratio and the stock volatility. The speculative demand is independent of age, which comes as no surprise given that its purpose is to achieve the best risk-return trade-off over the next trading interval, regardless of the horizon of the investor. By contrast, the hedging demand against Sharpe ratio risk depends explicitly upon age. Moreover, the fact that C_4 is increasing in the time-to-horizon implies that young investors will react more to a rise in the expected return than older ones.

[6] Under quite general assumptions, Wachter (2003) shows that the utility-maximising policy for an extremely risk-averse investor concerned with nominal terminal wealth is to fully invest in a nominal zero-coupon bond that matches his horizon.

[7] Kim and Omberg (1996) show that the functions $C_3(T-t;\gamma)$ and $C_4(T-t;\gamma)$ are positive and increasing for reasonable values of the parameter set.

It is unlikely that the allocation to the stock will be a monotonic decreasing function of age as advocated by current financial advice, since the Sharpe ratio and the volatility vary stochastically over time. Nevertheless, both C_3 and C_4 shrink to zero as the investor approaches his horizon date. As a consequence, an investor close to his horizon will only hold the speculative demand and cash.

2.3 Respecting the Solvency II requirements

The strategies that have been introduced in the previous section are designed to maximize expected utility from the terminal wealth at a given horizon. They are optimal in the long-term since they generate the highest possible average asset level for a given uncertainty in the terminal value of the assets. Nonetheless, such optimal strategies may end up violating short-term Solvency II constraints in market conditions with strong bear equity markets.

The focus of this section is to design a set of asset allocation strategies that should not only maximize the risk/return trade-off in terminal asset levels, but also satisfy given Solvency II budgets. The idea here is to have a separate control for risk-aversion, i.e. aversion with respect to uncertainty in long-term asset levels, and loss-aversion, i.e. aversion with respect to short-term losses. More precisely, the requirement here is that the wealth generated by the strategy at any time within a year should never fall below a fraction $1-\delta$ of the capital invested at the beginning of the year, where δ can be interpreted as a given Solvency II equity charge risk budget. This requirement can be mathematically formulated as :

$$A_t \geq (1-\delta) A_0, \quad \text{almost surely for all } t \text{ in } [0,\ 1Y] \tag{2.10}$$

Typical values for δ are 5%, 10%, 15%, and 20%. The *floor*, defined as $F = (1-\delta) A_0$ is the minimum acceptable wealth level, and the outstanding risk budget at time *t* is the distance of current wealth A_t to the floor.

In this context, and building on the comparison with constant proportion portfolio insurance (CPPI) strategies we propose to consider a family of strategies, which we label *Solvency II benchmarks* in what follows, where the weight allocated to stocks is of the form:

$$w_t^c = m\left(1 - \frac{F}{A_t^c}\right) w_t^* \tag{2.11}$$

Here w_t^* is the optimal weight referring to the unconstrained strategy of Proposition 2, m is a constant multiplier and A_t^c is the current value of the strategy. Note that in the limit of a floor converging to zero (no Solvency II capital charges), then the constrained allocation policy matches the unconstrained allocation policy, as it should.

Equation (2.11) says that the dollar amount allocated to the unconstrained strategy is an increasing function of the risk budget $A^c - F$. On the other hand, when the risk budget shrinks to zero, the portfolio is entirely invested in cash, since w_t^c is equal to zero. Since the cash investment grows faster than the floor, which is constant, a positive risk budget is recovered just after wealth has

landed on the floor. Hence strategy (2.11) guarantees that the wealth is always at least equal to the floor, and if the floor is attained, the portfolio does not remain invested in cash at later dates.

2.4 Implementation Challenges

This section discusses the practical implementation of the constrained w_t^c and unconstrained w_t^* strategies introduced in the previous part. These strategies are not directly implementable in practice, for the following three main reasons:

- They assume continuous trading, while trading is possible only in discrete time;
- They assume that short sales and leverage are permitted;
- They assume a perfect knowledge of the current Sharpe ratio and volatility of the stock.

To address the first two issues, we simply implement a discrete-time version of the strategy (2.11) with monthly trading, and resize the weights so as to rule out short positions and leverage. To address the third issue, we propose to use a parsimonious partition of the set of possible values for equity volatility and equity risk premium, and take robust estimates based on such partitions. It is common knowledge that the Sharpe ratio is much harder to estimate than the volatility, therefore we expect to have larger estimation errors coming from the Sharpe ratio estimation. Following Martellini and Milhau (2010), we propose to discretize the state space of these two processes, using a finer grid mesh for the volatility in order to take into account the difference in the estimation errors. This discretization introduces a loss of optimality that has been shown to remain low in Deguest et al. (2011). Let us now describe how these partitions are performed. Introducing upper and lower bounds for these two processes, the crudest possible partition of market conditions would involve replacing all realizations of the process λ^S by the constant $\lambda^S_{\text{mid}} = \frac{\lambda^S_{\text{inf}} + \lambda^S_{\text{sup}}}{2}$ (see section 3.3 for more details on how the equity risk premium process is filtered), and all realizations of σ^S by $\sigma^S_{\text{mid}} = \frac{\sigma^S_{\text{inf}} + \sigma^S_{\text{sup}}}{2}$.

The second partitioning level of the set of market conditions would involve distinguishing between high, moderate and low risk premium levels, and would therefore contain three buckets:

$$\left]-\infty, \frac{\lambda^S_{\text{inf}} + \lambda^S_{\text{mid}}}{2}\right], \quad \left]\frac{\lambda^S_{\text{inf}} + \lambda^S_{\text{mid}}}{2}, \frac{\lambda^S_{\text{mid}} + \lambda^S_{\text{sup}}}{2}\right], \quad \left]\frac{\lambda^S_{\text{mid}} + \lambda^S_{\text{sup}}}{2}, \infty\right[$$

and three standard values: λ^S_{inf}, λ^S_{mid} and λ^S_{sup}. More generally, it can be verified that the k_i^{th} ($k_i \geq 0$) partitioning level has $2^{k_i - 1} + 1$ standard values, where $i = 1$ for λ^S, and $i = 2$ for σ^S.

For the upper and lower bounds of λ^S and σ^S, we propose the following values

$$\lambda^S_{\text{inf}} = 0.25 \qquad \lambda^S_{\text{sup}} = 0.75$$
$$\sigma^S_{\text{inf}} = 0.05 \qquad \sigma^S_{\text{sup}} = 0.40,$$

Note that the volatility bounds are in line with observations of historical volatilities and of the VIX index for most of the available data. However, since the Sharpe ratio is not observable (see section 4.4 for more details on what exact crude proxy will be used for the unobservable Sharpe ratio on the

equity index), the choice of the bounds has to be done differently. We therefore propose to calibrate them using the parameters of the Sharpe ratio process. Indeed, they roughly correspond to the mean value $\overline{\lambda}$ plus or minus the long-term standard deviation $\sqrt{\frac{\sigma_\lambda}{2\kappa}}$ of λ^S. Once these bounds are set, we build a partition using a number of standard values (equally spread between the upper and lower bounds) equal to $2^{k_1-1}+1$ for λ^S and to $2^{k_2-1}+1$ for σ^S as explained previously. Since the estimation of the Sharpe ratio is known to be difficult, we will work with very low partitioning levels ($k_1 = 2$). Nonetheless, the estimation of the volatility is a pretty standard estimation, so we can consider high partitioning levels ($k_2 = 5$).

3. Calibration of the Model

In this section, we present the various approaches that we have considered in order to calibrate the model parameters. Table 1 summarizes the set of parameters that we will consider in our Monte Carlo simulations.

Table 1

Long-Term Parameter Values

This table displays the base case parameter values used in the numerical experiments.

Nominal Short-Term Rate	
b	3.06 %
a	0.13
σ^r	0.98%
λ^r	-53.00%
Sharpe Ratio	
κ	0.35
σ^λ	23.22%
$\bar{\lambda}$	40%
Volatility	
α	5.07
σ^V	48.00%
$\sqrt{\bar{V}}$	21.38%
Correlations	
$\rho^{S\lambda}$	– 1
ρ^{Sr}	0
ρ^{SV}	– 0.767
ρ^{rV}	0

3.1 Interest rate parameters: σ^r, λ^r, a and b

In order to calibrate the interest rate parameters, we use data from Bloomberg on the yield curve of the French Government. Then, we considered a maximum likelihood approach that has been extensively used in previous academic and practitioner studies, see for example by Fisher and Gilles (1996). The advantage of this method is that even for a single-factor Vasicek model, several yield maturities can be used in the calibration procedure. This method has been implemented in Duffee (2002) to calibrate more sophisticated models called "essentially affine". The main assumption is that we observe a zero-coupon bond yield without noise (we actually need here as many yields as the number of factors in the term structure model):

$$\tau_0 \tilde{Y}_t^{\tau_0} = D(\tau_0) r_t - a(\tau_0). \tag{3.1}$$

Then, we assume that all the other observations $\tilde{Y}_t^{\tau_i}$ with different maturities $\tau_1, \ldots, \tau_m$ will contain an additional measurement error $\varepsilon_t^{\tau_i} \sim N(0, \sigma_\varepsilon^2)$ like in the Kalman filter framework:

$$\tau_i \tilde{Y}_t^{\tau_i} = D(\tau_i) r_t - a(\tau_i) + \varepsilon_t^{\tau_i},$$

where r_t is equal to $\frac{\tau_0 Y_t^{\tau_0} + a(\tau_0)}{b_\alpha(\tau_0)}$ using equation (3.1). Since the measurement errors added to the other observations are assumed to be independent of each other and of the short rate r_t, the log-likelihood of the system is simply the sum of the log-likelihoods from the pure time series (equation (3.1)) and the pure cross-sectional estimations. If the distribution of the measurement errors is not

Gaussian, then a quasi maximum likelihood method can be used to compute an approximation of the objective function to maximize. Note that this approach has also been used to calibrate a dynamic asset allocation model in Sangvinatsos and Wachter (2005). The results of the calibration in table 1 are line with the literature (negative risk premium, low volatility).

3.2 Volatility parameters: σ^V, α and $\overline{V}$

The current value of volatility can be computed from daily returns of a stock index, either using the historical volatility formula or by fitting an EGARCH model. It can also be obtained from volatility indexes such as the VSTOXX index for the European market. We have favored the historical volatility because it is a simple estimator, and it can be computed from any stock returns, whereas existing volatility indexes only reflect the volatility of a given pool of stocks.

Concerning the parameters of the volatility dynamics, we use the calibration results of Ait-Sahalia and Kimmel (2007), where the authors extend the information of the historical returns of the stock index with the information available in option prices. They use the instantaneous volatility, proxied by the implied volatility, and then maximize the joint likelihood function of the observed data (S_t, V_t). This approach is called the integrated volatility proxy method, see Ait-Sahalia and Kimmel (2007).

Other estimation methods are discussed in the academic literature. Moment-based techniques are described by Chacko and Viceira (2003). The approach is based on the characteristic function of $\ln S_t$ which can be derived in closed form. A similar technique is discussed by Jiang and Knight (2002). Other methods include approximating the conditional moments of integrated volatility through high-frequency data, see Bollerslev and Zhou (2002), filtering techniques based on the characteristic function, see Bates (2006), and a Bayesian approach, see Eraker (2001).

A direct computation of the expected value of the VSTOXX index over the period 1999-2011 gives $\sqrt{\overline{V}} = 25.99\%$, and of its volatility (using adjusted volatility daily returns) gives $\sigma^V = 49.48\%$, which corroborates the results of Ait-Sahalia and Kimmel (2007) (see table 1), even though they use US market data. The parameters of the volatility process will not play any role in the strategy, but will only be used in the simulation procedure.

3.3 Sharpe ratio parameters: σ^λ, κ and $\overline{\lambda}$

In empirical tests, the challenge is to obtain estimates for both the current value of the equity Sharpe ratio, and also for the parameters that are involved in its diffusion process. In the empirical results presented in this paper, we follow an estimation approach based on the Solvency II methodology used to assign Capital Requirement for Equity (SCRE). Indeed, CEIOPS defines the SCRE as follows (see paragraph 3.88 page 49 of the calibration paper by CEIOPS (2010):

$$\text{SCRE} = 39\% + \max\left[\min\left(\underbrace{\frac{S_t - \overline{S_t}}{\overline{S_t}}}_{R_t};10\%\right);-10\%\right], \tag{3.2}$$

where S_t is the value at time t of the equity index, and

$$\overline{S_t} = \frac{1}{\sharp\text{days in 3Y}} \sum_{d_i=t-3Y}^{t-1\text{D}} S_{d_i}$$

is the moving average index value computed over the last 3Y of daily data. In other words, equation (3.2) means that we compute the return using the average stock index value of the last 3Y, denoted $\overline{S_t}$, and today's value S_t, and cap this value to 10% and floor it to -10%, and then add 39% to the final result. If the stock index has increased in value, the expected return looking forward is taken to be lower than before the increase in equity value, and therefore we increase the SCRE by at most 10%. Obviously, other approaches can be used as well. In particular, a standard approach in the academic literature, e.g. Viceira (2009), consists in using the Dividend Yield (or the Price/Earnings Ratio) to make the Sharpe ratio observable.

Regarding the parameters of the Sharpe ratio process, there is no clear consensus in the literature, since the Sharpe ratio is not observable. One reference paper using a mean-reverting process for the Sharpe ratio is Munk et al. (2004). The authors used a Kalman filter to calibrate their model (since they do not have a stochastic volatility component, their model remains Gaussian, and the Kalman filter can be used). The speed of mean reversion found in Munk et al. (2004) is equal to 6.08%, but their estimate comes with a high variance, which shows that this parameter is rather difficult to calibrate. The long-term mean of the Sharpe ratio is 44.10%, which is of the same order as the historical Sharpe ratio on the S&P500. Finally, they obtain a volatility for the Sharpe ratio equal to 4.70%.

Another approach to calibrate the parameters of the Sharpe ratio process could be to use the time series extracted from the dividend yield. Indeed, the linear regression of the stock index excess log-returns with respect to the log-dividend yield

$$\ln\left(\frac{S_{t+h}}{S_t}\right) - h \times Y_t^h = m\,DY_t + p + \varepsilon_t,$$

gives us the value of the two regression parameters m and p. Then, an estimate of the stock Sharpe ratio over the period $[t, t+h]$ can be obtained as follows from the expected excess log-returns:

$$h \times \left(\lambda_t^S \sigma_t^S - \tfrac{1}{2}\left(\sigma_t^S\right)^2\right) = m\,DY_t + p \quad \Rightarrow \quad \lambda_t^S = \frac{\frac{m\,DY_t + p}{h} + \frac{1}{2}\left(\sigma_t^S\right)^2}{\sigma_t^S} = \frac{\frac{m\,DY_t + p}{h} + \frac{1}{2}V_t}{\sqrt{V_t}}$$

The equation above relies on the fact that *h* is a short horizon, so the expected excess log-returns can be approximated by $\lambda_t^S \sigma_t^S - \frac{1}{2}\left(\sigma_t^S\right)^2$. Once we have extracted the Sharpe ratio time series, the calibration method relies on maximum likelihood estimation (MLE) technique, and allows one to estimate the three parameters of the Sharpe ratio process. The results of the MLE calibration using two different data sets are given in table 2.

Table 2

Calibration of the Equity Sharpe Ratio Parameters with MLE

This table displays the maximum likelihood estimates of the Sharpe ratio parameters using the time series extracted from the linear regression of the excess log-returns to the dividend yield, together with the historical volatility. Quarterly observations are used for both US and European datasets.

	S&P 500 (Mar 1989 - Mar 2011)	**EUROSTOXX 50 (Dec 1994 - Mar 2011)**
Calibrated Parameters		
$\bar{\lambda}$	44.25 %	36.45 %
κ	0.29	0.43
σ^{λ}	28.99 %	23.22 %
Model Implied Quantities		
$\frac{\sigma^{\lambda}}{\sqrt{2\kappa}}$	38.15 %	25.11 %

Since the estimates of κ, and $\bar{\lambda}$ are known to lack robustness, we have decided to be conservative and report the values of 0.35 for the speed of mean-reversion and 40% for the long-term Sharpe ratio. These values are two midpoints between 0.29 and 0.43, and 36.45% and 44.25% calibrated from the two different datasets. Note that the Solvency II benchmarks would be further magnified if one was to use a more optimistic value for the long-term Sharpe ratio. Nonetheless, the parameters obtained in both data sets are in line with the literature, e.g. Dimson et al. (2002).

3.4 Correlation parameters

The correlation parameters are set as follows. We take the correlation between unexpected stock returns and innovations to the Sharpe ratio to be -1. This choice makes Sharpe ratio risk spanned by the stock itself, and is supported by different calibrations on US market data: Campbell and Viceira (1999) report a correlation of -96% between excess returns on the CRSP value-weighted index and the dividend yield; Xia (2001) finds a value of -93%; and our own calibration yields a value of -98.3%. The correlation between unexpected stock returns and innovations to volatility is taken from Ait-Sahalia and Kimmel (2007) and therefore set to -76.7%. The negative value accounts for the fact that volatility tends to rise when stock returns go down. The other correlations are set to zero, since no significant correlation has been observed in the literature, e.g. Munk et al. (2004).

4. Numerical Analysis

In this section, we describe how we simulate the performance of the Solvency II benchmarks. In order to satisfy the Solvency II requirements, we need to impose short-term constraints at each time step. Since profits and losses are computed at the end of each calendar year, it seems natural to design the Solvency II benchmarks as roll-overs of strategies with a given horizon T, stopped after one year, and such that the Solvency II constraints described in section 2.4 are satisfied at each date. In other words, the benchmarks are reset on a yearly basis, with the arbitrary one-year time-frame chosen to be consistent with the Solvency II regulatory focus on short-term losses. We will let the horizon *T* vary across different values in the empirical illustration, in order to illustrate the properties of the strategy over short to long investment periods. Specifically, we will choose $T = 3Y, 5Y, 10Y, 15Y$.

The unconstrained strategy w^* is of the form given in proposition 2 (where we take the correlation ρ^{Sr} to be zero):

$$w_t^* = \frac{\lambda_t^S}{\gamma\sigma_t^S} - \left(1 - \frac{1}{\gamma}\right)\left[C_3(T-t;\gamma) + C_4(T-t;\gamma)\lambda_t^S\right]\beta_t^\lambda$$

leading to the Solvency II strategy given by:

$$w_t^c = m\left(1 - \frac{F}{A_t^c}\right)w_t^*$$

The process of weights w^c defines a family of Solvency II benchmark strategies for various values of the investment horizon $T = 3Y,\ 5Y,\ 10Y \text{ or } 15Y$ and various values of the Solvency II capital charge budget $\delta = 5,\ 10,\ 15 \text{ or } 20\%$. In order to take into account practical constraints, the implementation of these benchmarks will be done in discrete time, and will involve short-sale constraints on the unconstrained strategy w^*, and partitions of the state space of volatility and Sharpe ratio.

The family of Solvency II benchmarks is parameterized by two subjective parameters:

- the relative risk aversion γ;
- the constant multiplier m.

The risk aversion γ and multiplier values will be calibrated in such a way that the average allocation to the equity index is sufficiently high to obtain significant exposure to the equity risk premium, but also in such a way that the allocation never gets so high at any point in time that it might lead to violating the Solvency II floor in the empirical analysis.

In what follows, we consider two numerical experiments. The first one is based on a set of 10,000 Monte Carlo simulations using the parameters described in the first section. The second one is based on a first historical dataset, where we use the longest available official daily history (with a start in January 1997) for the RUSSELL Global Index (in US Dollars) as a proxy for the equity index, and the 3-month T-Bills as a proxy for the cash. We have also repeated the analysis with shorter historical dataset, where the RUSSELL Global and Developed indexes (currency-hedged version, in Euros,

available only since 1999, date of creation of the Euro) are used as proxies for the equity index, and the 1-month EURIBOR rate is used as a proxy for the cash, and obtained qualitatively similar results, which we do not report here for brevity. When using historical data, we report the distribution of the performance of the various Solvency II benchmarks over all one-year sub-periods in the sample period, with a starting date at the beginning of each calendar year.

4.1 Analysis based on Monte Carlo simulations

There are two key parameters of the model that we have not yet calibrated:

- the relative risk aversion γ
- the multiplier *m*

These two parameters are calibrated so as to generate a substantial exposure to the equity risk premium while ensuring that the Solvency II capital constraints are respected for all δ and *T*. The trade-off is as follows: since taking an exceedingly large value for the risk-aversion parameter γ will lead to the Solvency II benchmark being mostly invested in cash, we want to have a value for this parameter that generates a reasonably large average allocation to equities. Regarding the multiplier, increasing the *m* value will lead to increasing the upside potential (more aggressive spending of the risk budget) while also increasing the probability of hitting the floor or even underperforming the floor because of the presence of residual gap risk when the strategy is implemented in discrete time (here with a monthly frequency).

Our target was to obtain an average (across time and scenarios) allocation to equities that would range from around 20% to around 80% as a function of risk budgets δ and time-horizon *T*, with the lowest value corresponding to small risk budgets (i.e. δ = 5%) and short horizons (i.e. *T* = 3 years), and the highest value corresponding to large risk budgets (i.e. δ = 20%) and long horizons (i.e. *T* = 15 years). In what follows, we take the values $\gamma = 7$ and $m = 5$, which generate a range that is close to the target range (see table 3) without leading to violations of the risk budgets at the 99.5% confidence level.

We have performed a number of robustness checks, including testing that these outcomes (reasonably large range of allocations to equities without violations of the risk budgets) are robust with respect to perturbations of the current values of the interest rate, Sharpe ratio, and volatility. We have performed four different stress-tests (see table 11 of the Appendix for detailed results):

- low interest rate regime r_0=1%
- high volatility regime σ_0=25%
- low Sharpe ratio regime λ_0=25%
- combination of the three previous regimes

Once the choice of γ and *m* is fixed as above, we run 10,000 Monte Carlo simulations using the parameters described in table 1. The results in table 3 show that the average returns achieved by the Solvency II benchmarks are increasing in the capital charge δ, which was expected since the average

stock allocation increases also in the Solvency II risk budget δ. We also note that the wealth allocated to equities, and therefore the average performance, is also an increasing function of the time-to-horizon *T*, which can be explained by the decreasing term-structure of equity risk implied by the presence of mean-reversion in equity returns. Overall, we also note that even though the dynamic portfolio strategy is implemented in a discrete time (monthly), there is no violation of the target Solvency II risk budgets at the 99.5% confidence level, and in fact none at the 100% level given our scenarios. In fact, the risk budget is not entirely spent in most cases, and it is only for large δ and *T* values that the risk budgets are close to being spent. Hence, we find for *T* = 15 years a maximum loss of 19.58% for the δ = 20% risk budget (see panel C of table 3). In that sense, the Solvency II benchmarks achieve the initial objective, that is allows for a substantial allocation to equities while respecting given Solvency II risk budgets.

Table 3

1Y-Return Statistics and Risk Measures of the Solvency II Benchmarks

This table displays the performances and measures of risk of 16 Solvency II Benchmarks together with the average allocation in the simulated equity index over a 1Y-period.

Panel A: Horizon T=3Y

	δ=5%	δ=10%	δ=15%	δ=20%
Average Return	4.33 %	5.27 %	6.16 %	6.78 %
Standard Deviation of returns	2.73 %	4.81 %	6.72 %	7.92 %
Max Loss at 99.5%[1]	2.90 %	6.82 %	10.78 %	14.66 %
Max Loss[1]	3.57 %	8.12 %	12.77 %	17.42 %
Proba of Violating Floor	0 %	0 %	0 %	0 %
Av. Stock allocation	15.61 %	27.66 %	38.39 %	44.86 %

Panel B: Horizon T=5Y

	δ=5%	δ=10%	δ=15%	δ=20%
Average Return	4.57 %	5.69 %	6.72 %	7.44 %
Standard Deviation of returns	3.38 %	5.99 %	8.22 %	9.61 %
Max Loss at 99.5%[1]	3.36 %	7.59 %	11.91 %	16.20 %
Max Loss[1]	3.91 %	8.76 %	13.60 %	18.45 %
Proba of Violating Floor	0 %	0 %	0 %	0 %
Av. Stock allocation	19.09 %	33.65 %	45.93 %	53.33 %

Panel C: Horizon T=10Y

	δ=5%	δ=10%	δ=15%	δ=20%
Average Return	4.92 %	6.32 %	7.53 %	8.38 %
Standard Deviation of returns	4.53 %	7.96 %	10.64 %	12.30 %
Max Loss at 99.5%[1]	3.88 %	8.48 %	13.11 %	17.70 %
Max Loss[1]	4.45 %	9.31 %	14.30 %	19.29 %
Proba of Violating Floor	0 %	0 %	0 %	0 %
Av. Stock allocation	24.47 %	42.54 %	56.74 %	65.48 %

Panel D: Horizon T=15Y

	δ=5%	δ=10%	δ=15%	δ=20%
Average Return	5.08 %	6.57 %	7.85 %	8.76 %
Standard Deviation of returns	5.10 %	8.88 %	11.74 %	13.52 %
Max Loss at 99.5%[1]	4.04 %	8.73 %	13.46 %	18.19 %
Max Loss[1]	4.70 %	9.66 %	14.62 %	19.58 %
Proba of Violating Floor	0 %	0 %	0 %	0 %
Av. Stock allocation	26.78 %	46.19 %	61.09 %	70.35 %

(1) The max losses have been computed in percentage of the initial asset value A_0=100.

We now move on to compare the performance achieved by the Solvency II benchmarks with the performance achieved by comparable static benchmarks that involve a constant allocation to equities. We perform this comparison from two distinct dual perspectives.

First, for each value of δ and T, we compute the constant equity allocation such that the average returns of the static allocation match the ones of the SII benchmarks, and then analyze what the maximum losses are at the 99.5% and 100% confidence levels, and also what are the Solvency II capital charges that would correspond to these equity allocations.

Table 4 displays the performances of such static benchmarks, together with risk statistics. One important observation is that the max losses computed with a 99.5% probability are always higher than the tolerance threshold δ. In some cases, the maximum loss is more than twice the allocated risk budget (see for example in panel D of table 4 a maximum loss of 44.21% for the 15 year horizon and the 20% risk budget). Probabilities of violating the risk budget can also be substantial, and reach values as high as close to 7%. This shows that for a similar return performance, it is always better to choose the dynamic allocation offered by the SII benchmark in order to decrease the risk of violating the Solvency II constraints. Moreover, the use of the standard formula to compute the SII capital requirement leads to values that are higher than δ most of the time for *T*=5Y, and all the time for *T*=10Y and *T*=15Y.

Table 4

1Y-Return Statistics and Risk Measures of Static Allocations

This table displays the performances and measures of risk over a 1Y period of 16 static allocations strategies invested in the stock index and in cash. The allocation to equity is chosen such that the average returns of the static allocation match the average returns of the corresponding Solvency II Benchmarks of table 3. Solvency II Benchmarks together with measures of risk, and average allocation in the simulated equity index. The last row of each panel contains the capital requirement derived from the standard formula of the Solvency II regulation.

Panel A: Horizon T=3Y

	δ=5%	δ=10%	δ=15%	δ=20%
Average Return	4.33 %	5.27 %	6.16 %	6.79 %
Standard Deviation of returns	2.83 %	5.00 %	7.05 %	8.54 %
Max Loss at 99.5%[1]	6.04 %	12.18 %	17.57 %	21.24 %
Max Loss[1]	9.79 %	18.21 %	25.60 %	30.54 %
Proba of Violating Floor	1.05 %	1.24 %	1.11 %	0.76 %
Stock allocation	14.00 %	24.90 %	35.10 %	42.40 %
Capital Requirement (39% for equity)	5.46 %	9.71 %	13.69%	16.54%

Panel B: Horizon T=5Y

	δ=5%	δ=10%	δ=15%	δ=20%
Average Return	4.57 %	5.69 %	6.72 %	7.44 %
Standard Deviation of returns	3.38 %	5.96 %	8.38 %	10.08 %
Max Loss at 99.5%[1]	7.68 %	14.76 %	20.84 %	25.03 %
Max Loss[1]	11.93 %	21.76 %	30.01 %	35.32 %
Proba of Violating Floor	2.72 %	2.79 %	2.50 %	1.62 %
Stock allocation	16.80 %	29.70 %	41.60 %	49.90 %
Capital Requirement (39% for equity)	6.55 %	11.58 %	16.22%	19.46%

Panel C: Horizon T=10Y

	δ=5%	δ=10%	δ=15%	δ=20%
Average Return	4.92 %	6.32 %	7.53 %	8.38 %
Standard Deviation of returns	4.18 %	7.44 %	10.30 %	12.33 %
Max Loss at 99.5%[1]	9.94 %	18.53 %	25.58 %	30.24 %
Max Loss[1]	15.06 %	26.91 %	36.00 %	41.77 %
Proba of Violating Floor	5.41 %	5.54 %	4.98 %	3.48 %
Stock allocation	20.80 %	37.00 %	51.00 %	60.80 %
Capital Requirement (39% for equity)	8.11 %	14.43 %	19.89%	23.71%

Panel D: Horizon T=15Y

	δ=5%	δ=10%	δ=15%	δ=20%
Average Return	5.08 %	6.57 %	7.85 %	8.76 %
Standard Deviation of returns	4.56 %	8.03 %	11.07 %	13.24 %
Max Loss at 99.5%[1]	11.01 %	20.01 %	27.37 %	32.25 %
Max Loss[1]	16.53 %	28.88 %	38.23 %	44.21 %
Proba of Violating Floor	6.99 %	6.99 %	6.07 %	4.42 %
Stock allocation	22.70 %	39.90 %	54.70 %	65.20 %
Capital Requirement (39% for equity)	8.85 %	15.56 %	21.33%	25.43%

(1) The max losses have been computed in percentage of the initial asset value A_0=100.

Second, in order to have a better understanding of the opportunity costs involved in following standard static asset allocation strategies, as opposed to using dedicated dynamic asset allocation benchmarks that have been specifically engineered to allow for the optimal spending of the regulatory risk budgets, we turn to the dual analysis. More precisely, we consider the static benchmark that has an equity allocation leading to $\delta\%$ of SII capital requirement (using the standard formula of 39% for equity), and we then look at the corresponding average returns.

Table 5 displays the results obtained for such strategies. They show average returns that are lower than the returns of the 3Y-SII benchmark for $\delta = 5\%$, lower than the returns of the 5Y-SII benchmark for $\delta = 10\%$ and 15%, and lower than the returns of the 10Y-SII benchmark for $\delta = 20\%$. Overall, this analysis confirms that the static allocations do not perform as well as the SII benchmarks. Moreover, we see that the 99.5% max losses computed from our 10,000 Monte Carlo simulations are always higher than the capital requirement obtained from the SII standard formula, which suggests that lower stock allocations should be used, leading to even lower performances for the static benchmarks. We will observe the same results with the historical datasets, which illustrates that these results are not mere artifacts of our simulated scenarios.

Table 5

1Y-Return Statistics and Risk Measures of Static Allocations Respecting Solvency II

This table displays the performances and measures of risk over a 1Y period of 4 static allocations strategies invested in the stock index and in cash. The allocation to equity is chosen such that the capital requirement computed with the standard formula of the Solvency II regulation is equal to δ%

	δ=5%	δ=10%	δ=15%	δ=20%
Average Return	4.22 %	5.34 %	6.45 %	7.56 %
Standard Deviation of returns	2.60 %	5.15 %	7.74 %	10.36 %
Max Loss at 99.5%[1]	5.34 %	12.57 %	19.28 %	25.72 %
Max Loss[1]	8.88 %	18.76 %	27.91 %	36.17 %
Proba of Violating Floor	0.68 %	1.40 %	1.77 %	1.84 %
Stock allocation	12.82 %	25.64 %	38.46 %	51.28 %
Capital Requirement (39% for equity)	5 %	10 %	15%	20%

(1) The max losses have been computed in percentage of the initial asset value A_0=100.

Overall, these results suggest that the Solvency II benchmarks allow for a more efficient use of the Solvency II risk budget. Intuitively, this is because they enjoy the benefits of dynamic allocation strategies that can deliver the highest risk-adjusted long-term performance (the life-cycle investing paradigm) while respecting short-term risk budgets (the risk-controlled investing paradigm). In other words, the pre-commitment to reducing the allocation to equity in those times and market conditions that require such a reduction so as to avoid over-spending Solvency II risk budgets allows insurance companies to invest on average a higher allocation to equity compared to a simple static strategy that is calibrated so as to respect the same Solvency II constraints.

4.2 Analysis based on historical data

We now turn to the analysis based on historical data, which can be used to confirm whether the strong results obtained on the basis of Monte Carlo scenarios are confirmed in the context of actual return time-series. The backtests are performed using the Russell Global index in US Dollar as a proxy for the equity index, and the 3-month US T-Bills for cash.

Table 6 report the average annual performances obtained over each 1Y-period starting in January 1997 up to the end of 2010. The results show an average performance that increases with the maturity *T*. This observation is the same as for the Monte Carlo simulations, and illustrates again the downward-sloping term-structure for equities. The broad agreement of these results with the results obtained in the Monte Carlo analysis is a comforting indication of robustness of the results with respect to model and parameter specification.

Moreover, we observe that the average performances increase with δ, which was already the case in the Monte Carlo simulations. Of course, this result may not hold true for any historical scenario. In particular, when the dataset contains many falls in the equity market, increasing the risk budget will lead to lower, as opposed to higher, performance. It is only when an average performance is estimated over many different scenarios that increases in the risk budget will lead to increases in performance due to the positive equity risk premium.

The results based on historical scenarios also confirm that the Solvency II risk budgets are never over-spent (with a 99.5% confidence), even when looking at daily performance and when the sample period includes extremely severe bear markets as in 2000-2003 and 2008. In fact, even for the longest time-horizon Solvency II benchmarks of *T* = 15Y, the target risk budgets are respected over the sample period since the violation occurs less than 0.19% of the time, which is below 0.5%. Moreover, when a violation occurs, we note that the losses remain very close to the target of $\delta\%$. This last result suggests that Solvency II benchmarks can offer an extremely high level of safety to insurance companies, even though they do not come with the formal (and expensive) guarantee of option-based solutions.

Table 6

Backtest Results of the Solvency II Benchmarks

This table displays the average annual performance over each 1Y-period from January 1997 up to the end of 2010. The dataset includes the Russell Global equity index in US Dollars and the 3-month US T-Bills. The allocation is rebalanced on a monthly basis.

Panel A: Horizon T=3Y

	δ=5%	δ=10%	δ=15%	δ=20%
Average Performance	4.81 %	6.25 %	7.01 %	7.23 %
Max Loss at 99.5%[1]	4.24 %	8.75 %	13.25 %	17.75 %
Max Loss[1]	4.69 %	9.51 %	14.32 %	19.13 %
Proba of Violating Floor	0 %	0 %	0 %	0 %

Panel B: Horizon T=5Y

	δ=5%	δ=10%	δ=15%	δ=20%
Average Performance	5.20 %	6.92 %	7.88 %	8.26 %
Max Loss at 99.5%[1]	4.57 %	9.32 %	14.07 %	18.82 %
Max Loss[1]	4.99 %	10.02 %	15.05 %	20.08 %
Proba of Violating Floor	0 %	0.03 %	0.03 %	0.03 %

Panel C: Horizon T=10Y

	δ=5%	δ=10%	δ=15%	δ=20%
Average Performance	5.52 %	7.54 %	8.81 %	9.37 %
Max Loss at 99.5%[1]	4.81 %	9.72 %	14.64 %	19.55 %
Max Loss[1]	5.23 %	10.41 %	15.59 %	20.77 %
Proba of Violating Floor	0.08 %	0.08 %	0.08 %	0.08 %

Panel D: Horizon T=15Y

	δ=5%	δ=10%	δ=15%	δ=20%
Average Performance	5.54 %	7.57 %	8.96 %	9.58 %
Max Loss at 99.5%[1]	4.87 %	9.82 %	14.77 %	19.72 %
Max Loss[1]	5.33 %	10.57 %	15.81 %	21.04 %
Proba of Violating Floor	0.14 %	0.16 %	0.19 %	0.19 %

(1) The max losses have been computed in percentage of the initial asset value A_0=100, and on a daily basis.

4.3 Impact of tracking error constraints for active portfolio management

In practice, the allocation to equities by insurance companies can take the form of investment in active managers. This is a possible source of concern since the tracking error of the active manager return will imply a deviation of the portfolio return from the passive index returns, which are used to generate the Solvency II benchmark allocations. In this context, one might wonder what the maximum level of tracking error would be so as to maintain the absence of violation of the risk budgets.

- Moderate level of active portfolio management with an average α = 1% added value and a tracking error *TE* = 2%,
- High level of active portfolio management with an average α = 2% added value and a tracking error *TE* = 4%.

The methodology we use consists in modeling the added-value by the active manager as a process that is independent from the return of the equity index (an assumption which will be relaxed below), and has a mean and standard deviation that are consistent with the assumptions made on α and *TE*. More precisely, the new return of the equity index is now modeled as:

$$r^S_{t,t+h} \rightarrow r^S_{t,t+h} + \alpha h + TE\sqrt{h}\varepsilon_{t,t+h,} \quad \text{where} \quad \varepsilon_{t,t+h} \sim N(0,1)$$

Therefore, in order to obtain a benchmark with active management, we use the same weights as for the benchmarks with passive management (i.e. when α = *TE* = 0) but invest in an actively managed equity index. Table 7 covers the case of moderately active portfolio management, while table 8 exhibits the case of a high level of active management.

It is interesting to note that the active management of the portfolio has a small negative influence on the max losses, but does not have a negative impact on the max losses with a 99.5% probability. In other words, only the minimum of the portfolio distribution is found to be slightly affected by high tracking error values, but not the quantile at a 99.5% level. In all cases, the maximum loss remains within the risk budget. On the other hand, the benefit from a positive alpha leads to a substantial increase in average performance.

Table 7

Backtest Results of the Solvency II Benchmarks with Moderate Portfolio Management

This table displays the average annual performance over each 1Y-period from January 1997 up to the end of 2010. The dataset includes the Russell Global equity index in US Dollars (with moderate active portfolio management) and the 3-month US T-Bills. The moderate level of active portfolio management is simulated with an average α = 1% added value and a tracking error *TE* = 2%.

Panel A: Horizon T=3Y

	δ=5%	δ=10%	δ=15%	δ=20%
Average Performance	5.05 %	6.75 %	7.72 %	8.05 %
Max Loss at 99.5%[1]	3.90 %	8.13 %	12.39 %	16.65 %
Max Loss[1]	4.43 %	9.06 %	13.68 %	18.33 %
Proba of Violating Floor	0 %	0 %	0 %	0 %

Panel B: Horizon T=5Y

	δ=5%	δ=10%	δ=15%	δ=20%
Average Performance	5.53 %	7.59 %	8.79 %	9.32 %
Max Loss at 99.5%[1]	4.18 %	8.61 %	13.07 %	17.55 %
Max Loss[1]	4.58 %	9.28 %	14.01 %	18.77 %
Proba of Violating Floor	0 %	0 %	0 %	0 %

Panel C: Horizon T=10Y

	δ=5%	δ=10%	δ=15%	δ=20%
Average Performance	5.91 %	8.31 %	9.86 %	10.62 %
Max Loss at 99.5%[1]	4.41 %	8.99 %	13.61 %	18.25 %
Max Loss[1]	4.81 %	9.66 %	14.54 %	19.44 %
Proba of Violating Floor	0 %	0 %	0 %	0 %

Panel D: Horizon T=15Y

	δ=5%	δ=10%	δ=15%	δ=20%
Average Performance	5.95 %	8.36 %	10.06 %	10.89 %
Max Loss at 99.5%[1]	4.45 %	9.07 %	13.71 %	18.38 %
Max Loss[1]	4.90 %	9.80 %	14.73 %	19.68 %
Proba of Violating Floor	0 %	0 %	0 %	0 %

(1) The max losses have been computed in percentage of the initial asset value A_0=100, and on a daily basis.

Table 8

Backtest Results of the Solvency II Benchmarks with High Portfolio Management

This table displays the average annual performance over each 1Y-period from January 1997 up to the end of 2010. The dataset includes the Russell Global equity index in US Dollars (with high active portfolio management) and the 3-month US T-Bills. The high level of active portfolio management is simulated with an average α = 2% added value and a tracking error *TE* = 4%.

Panel A: Horizon T=3Y

	δ=5%	δ=10%	δ=15%	δ=20%
Average Performance	5.30 %	7.26 %	8.43 %	8.87 %
Max Loss at 99.5%[1]	3.65 %	7.84 %	12.06 %	16.23 %
Max Loss[1]	4.68 %	9.54 %	14.35 %	19.09 %
Proba of Violating Floor	0 %	0 %	0 %	0 %

Panel B: Horizon T=5Y

	δ=5%	δ=10%	δ=15%	δ=20%
Average Performance	5.86 %	8.25 %	9.71 %	10.40 %
Max Loss at 99.5%[1]	3.84 %	8.13 %	12.51 %	16.77 %
Max Loss[1]	4.68 %	9.54 %	14.42 %	19.25 %
Proba of Violating Floor	0 %	0 %	0 %	0 %

Panel C: Horizon T=10Y

	δ=5%	δ=10%	δ=15%	δ=20%
Average Performance	6.31 %	9.09 %	10.93 %	11.90 %
Max Loss at 99.5%[1]	4.06 %	8.59 %	13.10 %	17.52 %
Max Loss[1]	4.73 %	9.82 %	14.87 %	19.82 %
Proba of Violating Floor	0 %	0 %	0 %	0 %

Panel D: Horizon T=15Y

	δ=5%	δ=10%	δ=15%	δ=20%
Average Performance	6.35 %	9.15 %	11.17 %	12.23 %
Max Loss at 99.5%[1]	4.14 %	8.73 %	13.32 %	17.80 %
Max Loss[1]	4.80 %	9.95 %	15.06 %	20.06 %
Proba of Violating Floor	0 %	0 %	0.03 %	0.03 %

(1) The max losses have been computed in percentage of the initial asset value A_0=100, and on a daily basis.

In order to assess how large the tracking error can be before generating violations of the risk budgets, we set α = 0% so as to neutralize the benefits of active portfolio management in terms of expected performance, and increase the tracking error *TE* until we see the first constraint violations. The results of this numerical experiment give a tracking error of 6%. In that case, the violations of the target risk budgets with a 99.5% probability are very small (15.05% for δ = 15% and 20.05% for δ = 20%), which shows that our benchmarks can be implemented with actively managed equity proxies. Our experiment also reveals that for *TE* = 6%; the violations disappear as soon as $\alpha \geq 1\%$. Obviously, in investment practice, tracking error is not as well behaved as in our controlled experiment. In particular, one expects the ex-post realized tracking error to vary in time, and to exhibit extreme values in some market conditions. Our results suggest that an efficient control of tracking error that could maintain the Value-at-Tracking Error Risk (VaTER) around 6% would allow insurance companies to use the Solvency II benchmarks with actively managed equity proxies. While higher levels of tracking error would likely generate violations of the risk budget, such violations are not likely to be large in magnitude and frequent in probability.

While pure added-value by active managers can naturally be represented by a process that is uncorrelated with the underlying benchmark, it can happen that the excess return over the benchmark has a positive correlation with the benchmark itself. This situation is intuitively less attractive than the zero correlation case, since it implies that negative outperformance of the active manager is to be expected on average when the underlying index performs poorly. To test for the impact of this feature on possible violations of the risk budget, we simulate in what follows the case of an active manager who generates excess returns that have a non-zero correlation, denoted by ρ, with the index. This leads to the following model for the actively managed returns:

$$r^S_{t,t+h} \rightarrow r^S_{t,t+h} + \alpha h + TE\sqrt{h}\left[\rho\left(\frac{r^S_{t,t+h} - \overline{r^S_{t,t+h}}}{\sigma^S_t\sqrt{h}}\right) + \sqrt{1-\rho^2}\,\varepsilon_{t,t+h}\right]$$

$$\text{where } \varepsilon_{t,t+h} \sim N(0,1) \text{ is independent from } r^S_{t,t+h}, \text{ and } \overline{r^S_{t,t+h}} = \sum_{i=1}^{n} r^S_{t_i,t_i+h}$$

We can easily note that for $\rho = 0$, the returns have the same expression as before, i.e. when no correlation was modeled in the active management. In general, when $\rho \neq 0$, we can also check that the added value by the manager has an annualized mean equal to α and a tracking error equal to TE, as before.

Then, we use the weights computed with passive management (i.e. when α = *TE* = 0) and invest in an actively managed equity index. Table 9 below presents the case of an investment in a highly active manager (i.e. when α =2%, and *TE* = 4%) whose added value process exhibits a substantial positive correlation with the Global Russell equity index (ρ =50%). For simplicity, we focus on T=15 years, where the effects are likely to be most pronounced, since this is the situation where the investment in equities is the highest.

Table 9

Backtest Results of the 15Y-Solvency II Benchmarks with High Correlated Portfolio Management

This table displays the average annual performance over each 1Y-period from January 1997 up to the end of 2010. The dataset includes the Russell Global equity index in US Dollars (with high active portfolio management) and the 3-month US T-Bills. The high level of active portfolio management is simulated with an average α = 2% added value and a tracking error *TE* = 4%. The added value is also simulated with a ρ =50% correlation with the stock index returns.

	δ=5%	δ=10%	δ=15%	δ=20%
Average Performance	6.95 %	10.24 %	12.62 %	13.85 %
Max Loss at 99.5%[1]	4.89 %	9.86 %	14.85 %	19.83 %
Max Loss[1]	5.26 %	10.57 %	15.93 %	21.12 %
Proba of Violating Floor	0.11 %	0.14 %	0.19 %	0.16 %

(1) The max losses have been computed in percentage of the initial asset value A_0=100, and on a daily basis.

As expected, we find that the max losses have also increased compared to the case with no correlation. This is explained by the fact that severe drawdowns of equity markets are amplified by the actively managed fund in the presence of a positive correlation ρ. On the other hand, we find that at the 99.5% level losses are still below the target risk budgets δ. We also find a small positive impact on the average performances since they increase

Finally, we have maintained the assumption of a high correlation $\rho = 50\%$, set α = 0% so as to neutralize the benefits of active portfolio management in terms of expected performance, and increased the tracking error *TE* until we see the first constraint violations. The results of this numerical experiment give a tracking error of 2%, to be compared with a tracking error of 6% when the correlation was zero.

Table 10

Backtest Results of the 15Y-Solvency II Benchmarks with Poor Correlated Portfolio Management

This table displays the average annual performance over each 1Y-period from January 1997 up to the end of 2010. The dataset includes the Russell Global equity index in US Dollars (with poor active portfolio management) and the 3-month US T-Bills. The poor level of active portfolio management is simulated with an average α = 0% added value and a tracking error *TE* = 2%. The added value is also simulated with a ρ =50% correlation with the stock index returns.

	δ=5%	δ=10%	δ=15%	δ=20%
Average Performance	5.91 %	8.30 %	9.96 %	10.74 %
Max Loss at 99.5%[1]	5.01 %	10.07 %	15.13 %	20.19 %
Max Loss[1]	5.43 %	10.74 %	16.05 %	21.36 %
Proba of Violating Floor	0.71 %	1.37 %	1.45 %	1.51 %

(1) The max losses have been computed in percentage of the initial asset value A_0=100, and on a daily basis.

From table 10, we can note that active management that is highly correlated with the stock returns tends to amplify the max losses of the original index (passively managed), and needs therefore to exhibit high values of α in order to offset the negative impact of the correlation. These results suggest that the ability to identify active managers that are likely to generate a stable level of tracking error and also to generate pure alpha, as opposed to correlated abnormal performance, is a key source of added value when attempting to implement the Solvency II benchmarks with non-passive vehicles.

5. Conclusion

The Solvency II framework for regulatory capital makes investment in equities very costly. As a consequence of the high cost of investing in equity from a solvency capital requirement perspective, it is believed that insurance companies will almost exclusively end up investing in fixed-income securities, with a substantial associated opportunity cost related to almost entirely giving up on the equity risk premium. The purpose of this paper is to introduce a methodological framework to allow insurance companies to gain exposure to equity markets, and thus benefit from the associated risk premium, while maintaining their solvency capital expenditure at reasonable levels.

From a conceptual standpoint, the dynamic investment strategy taking into account solvency constraints can be regarded as the replicating portfolio strategies for a contingent claim written on the underlying unconstrained strategy, in such a way that the underlying strategy is recovered as a specific case in the limit of vanishing solvency capital requirements. The design of the optimal investment strategy for an insurance company in the absence of short-term solvency constraints should therefore be the natural starting point. By continuity, any optimal investment solution in the presence of solvency constraints should formally converge to the optimal unconstrained long-term solution in the limit of vanishing solvency constraints.

In implementation, we consider solvency constraints taken to be a series of monthly budget constraints, with a 12-month periodicity that is somewhat arbitrary, but that corresponds to the regulatory framework. The results obtained from numerical analysis based on both stochastic simulations and historical track records, confirm that Solvency II dynamic asset allocation benchmarks can be designed so as to allow for a more efficient use of the Solvency II risk budget. Intuitively, this is because the pre-commitment to reduce the allocation to equity in times and market conditions that require such a reduction so as to avoid over-spending Solvency II risk budgets, allows insurance companies to invest on average more into equities compared to a simple static strategy that is calibrated so as to respect the same Solvency II constraints. The welfare gains involved are found to be substantial for reasonable parameter values, and are relatively robust with respect to implementation constraints and the presence of tracking error risk.

Appendix

Proof of proposition 1

The static problem obtained with the minimax pricing kernel reads:

$$\max_{A_T} E\left[U\left(A_T\right)\right], \quad \text{s.t. } E\left[M_T A_T\right] = A_0 \tag{A.1}$$

The optimal payoff in (A.1) is obtained by writing the first-order optimality condition:

$$A_T^* = \frac{A_0}{E\left[\left(M_T^*\right)^{1-\frac{1}{\gamma}}\right]}\left(M_T^*\right)^{-\frac{1}{\gamma}}$$

The time t-value of the optimal portfolio is then obtained by discounting the optimal payoff with the minimax pricing kernel:

$$A_t^* = E_t\left[\frac{M_T^* A_T^*}{M_t^*}\right] = \frac{A_0}{E\left[\left(M_T^*\right)^{1-\frac{1}{\gamma}}\right]}\left(M_t^*\right)^{-\frac{1}{\gamma}} E_t\left[\left(\frac{M_T^*}{M_t^*}\right)^{1-\frac{1}{\gamma}}\right] \tag{A.2}$$

It is reasonable to assume that the conditional expectation on the right side can only be written as a function of time, current interest rate and Sharpe ratio:

$$E_t\left[\left(\frac{M_T^*}{M_t^*}\right)^{1-\frac{1}{\gamma}}\right] = g(t, r_t, \lambda_t^S)$$

It remains to find the form for the function g. The process $M^* A^*$ is a martingale, so its drift term must be zero:

$$\begin{aligned} 0 = {} & \frac{1-\gamma}{\gamma} r_t + \frac{1-\gamma}{2\gamma^2}\left\|\underline{\lambda}_t + \underline{\nu}_t^*\right\|^2 + \frac{g_t}{g} + \frac{g_r}{g} a\left(b - r_t\right) + \frac{g_\lambda}{g}\kappa\left(\overline{\lambda} - \lambda_t^S\right) + \frac{g_{rr}}{2g}\left(\sigma^r\right)^2 + \frac{g_{\lambda\lambda}}{2g}\left(\sigma^\lambda\right)^2 \\ & + \frac{g_{r\lambda}}{g}\rho^{r\lambda}\sigma^\lambda\sigma^r + \frac{1-\gamma}{\gamma}\left(\underline{\lambda}_t + \underline{\nu}_t^*\right)'\left[\frac{g_r}{g}\sigma^r\underline{\rho}^r + \frac{g_\lambda}{g}\sigma^\lambda\underline{\rho}^\lambda\right] \end{aligned} \tag{A.3}$$

Moreover, the volatility vector of A^* must be parallel to the unit volatility of the stock index $\underline{\rho}^S$. Introducing the matrix $\underline{\underline{N}} = \underline{\underline{I}}_4 - \underline{\rho}^S\left(\underline{\rho}^S\right)'$ and applying Ito's lemma to (A.2), we can write this condition as:

$$\underline{\underline{N}}\left[\frac{1}{\gamma}\left(\underline{\lambda}_t + \underline{\nu}_t^*\right) + \frac{g_r}{g}\sigma^r\underline{\rho}^r + \frac{g_\lambda}{g}\sigma^\lambda\underline{\rho}^\lambda\right] = 0$$

Since $\underline{\underline{N}}\underline{\lambda}_t = 0$ and $\underline{\underline{N}}\underline{\nu}_t^* = \underline{\nu}_t^*$, this is equivalent to:

$$\nu_t^* = -\gamma\frac{g_r}{g}\sigma^r\underline{\underline{N}}\underline{\rho}^r - \gamma\frac{g_\lambda}{g}\sigma^\lambda\underline{\underline{N}}\underline{\rho}^\lambda$$

Substituting this back into (A.3), we obtain that g must be a solution to the following partial differential equation (PDE):

$$
\begin{aligned}
0 = & \frac{1-\gamma}{\gamma} r_t + \frac{g_t}{g} + \frac{g_r}{g} a(b-r) + \frac{g_\lambda}{g} \kappa\left(\overline{\lambda} - \lambda\right) + \frac{g_{rr}}{2g}\left(\sigma^r\right)^2 + \frac{g_{\lambda\lambda}}{2g}\left(\sigma^\lambda\right)^2 + \frac{g_{r\lambda}}{2g}\rho^{r\lambda}\sigma^r\sigma^\lambda \\
& + \frac{1-\gamma}{2\gamma^2}\left\| -\lambda \underline{\rho}^S + \gamma \frac{g_r}{g}\sigma^r \underline{\underline{N}}\underline{\rho}^r + \gamma \frac{g_\lambda}{g}\sigma^\lambda \underline{\underline{N}}\underline{\rho}^\lambda \right\|^2 \\
& + \frac{1-\gamma}{\gamma}\left[\lambda \underline{\rho}^S - \gamma \frac{g_r}{g}\sigma^r \underline{\underline{N}}\underline{\rho}^r - \gamma \frac{g_\lambda}{g}\sigma^\lambda \underline{\underline{N}}\underline{\rho}^\lambda \right]' \left[\frac{g_r}{g}\sigma^r \underline{\rho}^r + \frac{g_\lambda}{g}\sigma^\lambda \underline{\rho}^\lambda \right]
\end{aligned}
\tag{A.4}
$$

The terminal condition is $g(T,r,\lambda)=1$. Conversely, if g solves PDE (A.4) with that terminal condition, then the process $\left(M_t^*\right)^{1-\frac{1}{\gamma}} g\left(t, r_t, \lambda_t^S\right)$ follows a martingale, so that $g\left(t, r_t, \lambda_t^S\right)$ is indeed equal to the conditional expectation at date t of $\left[M_T^* / M_t^*\right]^{1-\frac{1}{\gamma}}$. This justifies *ex-post* our conjecture regarding the functional form of that conditional expectation.

We now solve (A.4) for the function g. Given the affine structure of the model, a qualified guess for g is the exponential of a function affine in r and quadratic in λ (see Liu (2007):

$$
g(t,r,\lambda) = \exp\left[\frac{1-\gamma}{\gamma}\left[C_1(T-t;\gamma) + C_2(T-t;\gamma) r + C_3(T-t;\gamma)\lambda + \frac{1}{2} C_4(T-t;\gamma)\lambda^2 \right]\right]
$$

If we substitute back the first- and second-order partial derivatives of g into the right side of (A.4), we obtain a linear combination of 1, r, λ and λ^2 with deterministically time-dependent coefficients. This linear combination must be zero for each value of r and λ, hence the four time-dependent coefficients must be zero at all dates. Writing each of these conditions leads to the following system of four coupled ordinary differential equations (ODEs):

$$
\begin{aligned}
\frac{\partial C_4}{\partial u} &= \frac{1}{\gamma} - 2\overline{\kappa} C_4 + \frac{1-\gamma}{\gamma}\left(\sigma^\lambda\right)^2\left[\gamma + (1-\gamma)\left(\rho^{S\lambda}\right)^2\right] C_4^2 \\
\frac{\partial C_3}{\partial u} &= \kappa\overline{\lambda} C_4 - \overline{\kappa} C_3 + \frac{1-\gamma}{\gamma}\rho^{Sr}\sigma^r C_2 + \frac{1-\gamma}{\gamma}\left(\sigma^\lambda\right)^2\left[\gamma + (1-\gamma)\left(\rho^{S\lambda}\right)^2\right] C_3 C_4 \\
&\quad + \frac{1-\gamma}{\gamma}\sigma^r\sigma^\lambda\left[\gamma\rho^{r\lambda} + (1-\gamma)\rho^{Sr}\rho^{S\lambda}\right] C_2 C_4 \\
\frac{\partial C_2}{\partial u} &= 1 - a C_2 \\
\frac{\partial C_1}{\partial u} &= abC_2 + \kappa\overline{\lambda} C_3 + \frac{1}{2}\left(\sigma^\lambda\right)^2 C_4 + \frac{1-\gamma}{2\gamma}\left(\sigma^r\right)^2\left[\gamma + (1-\gamma)\left(\rho^{Sr}\right)^2\right] C_2^2 \\
&\quad + \frac{1-\gamma}{2\gamma}\left(\sigma^\lambda\right)^2\left[\gamma + (1-\gamma)\left(\rho^{S\lambda}\right)^2\right] C_3^2 + \frac{1-\gamma}{\gamma}\sigma^r\sigma^\lambda\left[\gamma\rho^{r\lambda} + (1-\gamma)\rho^{Sr}\rho^{S\lambda}\right] C_2 C_3
\end{aligned}
\tag{A.5}
$$

In these equations, we have set $\overline{\kappa} = \kappa - \frac{1-\gamma}{\gamma}\rho^{S\lambda}\sigma^\lambda$.

The optimal portfolio policy is given by:

$$w_t^* = \frac{1}{\sigma_t^S}\left(\underline{\rho}^S\right)' \sigma_t^{A^*}$$

where $\sigma_t^{A^*}$ is the volatility vector of A^*. Applying Ito's lemma in equation (A.2), we get that:

$$w_t^* = \frac{\lambda_t^S}{\gamma\sigma_t^S} - \left(1-\frac{1}{\gamma}\right)\frac{\rho^{Sr}\sigma^r}{\sigma_t^S} C_2\left(T-t;\gamma\right) - \left(1-\frac{1}{\gamma}\right)\frac{\rho^{S\lambda}\sigma^\lambda}{\sigma_t^S}\left[C_3\left(T-t;\gamma\right) + C_4\left(T-t;\gamma\right)\lambda_t^S\right]$$

The indirect utility function is defined by:

$$J(t, A_t^*, r_t, \lambda_t^S) = \frac{1}{1-\gamma} E_t\left[\left(A_T^*\right)^{1-\gamma}\right]$$

Substituting the expression of the optimal payoff, we get that:

$$\begin{aligned} J\left(t, A_t^*, r_t, \lambda_t^S\right) &= \frac{\eta^{1-\frac{1}{\gamma}}}{1-\gamma} E_t\left[\left(M_T^*\right)^{1-\frac{1}{\gamma}}\right] \\ &= \frac{1}{1-\gamma}\left(A_t^*\right)^{1-\gamma} g\left(t, r_t, \lambda_t^S\right)^\gamma \end{aligned}$$

Table 11

Stress-test Results of the Solvency II Benchmarks

This table displays the performance and measure of risk of 16 Solvency II Benchmarks over a 1Y-period. Each of the first three panels corresponds to a different stress-test regime. The last panel combines all the different regimes together.

Panel A: Low interest rate regime r_0=1%

		δ=5%	δ=10%	δ=15%	δ=20%
T=3Y	Average Return	2.19 %	3.12 %	4.03 %	4.70 %
	Max Loss at 99.5%	3.56 %	7.53 %	11.52 %	15.50 %
T=5Y	Average Return	2.39 %	3.50 %	4.56 %	5.32 %
	Max Loss at 99.5%	3.92 %	8.23 %	12.55 %	16.86 %
T=10Y	Average Return	2.69 %	4.07 %	5.32 %	6.22 %
	Max Loss at 99.5%	4.34 %	8.95 %	13.57 %	18.16 %
T=15Y	Average Return	2.81 %	4.31 %	5.63 %	6.59 %
	Max Loss at 99.5%	4.46 %	9.16 %	13.90 %	18.61 %

Panel B: High volatility regime σ_0=25%

		δ=5%	δ=10%	δ=15%	δ=20%
T=3Y	Average Return	4.31 %	5.23 %	6.10 %	6.71 %
	Max Loss at 99.5%	2.91 %	6.83 %	10.79 %	14.73 %
T=5Y	Average Return	4.54 %	5.64 %	6.65 %	7.36 %
	Max Loss at 99.5%	3.36 %	7.62 %	11.93 %	16.17 %
T=10Y	Average Return	4.89 %	6.26 %	7.44 %	8.28 %
	Max Loss at 99.5%	3.86 %	8.47 %	13.09 %	17.70 %
T=15Y	Average Return	5.04 %	6.51 %	7.77 %	8.66 %
	Max Loss at 99.5%	4.04 %	8.71 %	13.44 %	18.19 %

Panel C: Low Sharpe ratio regime λ_0=25%

		δ=5%	δ=10%	δ=15%	δ=20%
T=3Y	Average Return	3.76 %	4.25 %	4.74 %	5.11 %
	Max Loss at 99.5%	2.46 %	6.11 %	9.79 %	13.49 %
T=5Y	Average Return	3.90 %	4.50 %	5.09 %	5.53 %
	Max Loss at 99.5%	2.99 %	7.09 %	11.13 %	15.16 %
T=10Y	Average Return	4.13 %	4.89 %	5.63 %	6.16 %
	Max Loss at 99.5%	3.61 %	8.06 %	12.51 %	16.99 %
T=15Y	Average Return	4.22 %	5.06 %	5.85 %	6.44 %
	Max Loss at 99.5%	3.82 %	8.42 %	13.01 %	17.57 %

Panel D: Combination of the 3 regimes

		δ=5%	δ=10%	δ=15%	δ=20%
T=3Y	Average Return	1.69 %	2.17 %	2.65 %	3.05 %
	Max Loss at 99.5%	3.26 %	6.93 %	10.62 %	14.32 %
T=5Y	Average Return	1.81 %	2.39 %	2.98 %	3.44 %
	Max Loss at 99.5%	3.67 %	7.75 %	11.81 %	15.85 %
T=10Y	Average Return	1.99 %	2.73 %	3.47 %	4.04 %
	Max Loss at 99.5%	4.13 %	8.62 %	13.07 %	17.52 %
T=15Y	Average Return	2.07 %	2.88 %	3.68 %	4.29 %
	Max Loss at 99.5%	4.28 %	8.89 %	13.48 %	18.09

(1) The max losses have been computed in percentage of the initial asset value A_0=100, and on a daily basis.

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