# A New Convexity Adjustment for a CMS under a Multi-Curve Framework

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**Abstract**

The crisis that affected financial markets in the last years, leaded market practitioners to revise well known basic concepts like the ones of discount factors and forward rates. A single yield curve is not sufficient any longer to describe the market of interest rate products. On the other hand, using different yield curves at the same time, requires a reformulation of most of the basic assumptions made.

We investigate how this approach of using different yield curves for discounting and forwarding is used to price a specific interest rate product, the Constant Maturity Swap (CMS). CMS are money market instruments whose valuation requires the use of a convexity adjustment. The standard convexity adjustment is derived under the assumption that the term structure of interest rates is flat and has only parallel shifts. We develop a new convexity adjustment for the case when the term structure may tilt, under the new framework of different curves. We calibrate CMS spreads to market data and compare our adjustments with the SABR CMS adjustment widely used in the market.

# 1 Introduction

Since the beginning of the financial crisis in August 2007, the money markets have exhibited unprecedented behaviour. Interest rates, like for instance OIS and deposit rates with the same maturity, or swap rates with the same tenor but based on different floating-leg frequencies, that previously had been following each other closely for a long time, have started to keep a significant distance apart. Another example is the Forward Rate Agreement (FRA), which before crisis, could be replicated by long and short positions in two deposits, with the implied forward rate differing only slightly from the corresponding quantity obtained through OIS rates, now is quoted with large non-negligible spreads. This divergence in values does not create arbitrage opportunities when credit or liquidity issues are taken into account, which means that the market is pricing in other risks such as counterparty and liquidity risks.

In this paper, we will assume that the discount curve coincides with that stripped from OIS swap rates. Since OIS rates can be regarded as the best available proxy for risk neutral rates, this is equivalent to assume zero counterparty risk in the market plain vanilla instruments. Since an overnight rate refers to lending for an extremely short period of time, it is assumed to incorporate negligible credit or liquidity risk. Furthermore, there is very little default risk in the OIS market because there is no exchange of principal; funds are exchanged only at the maturity of the contract, when one party pays the net interest obligation to the other.

Before the onset of the turmoil in the credit markets in August 2007, the Libor-OIS spread was so little (below 10 bp) that it was acceptable consider both quotes risk free. When the spread grows, OIS is definitely a better approximation for a riskless rate, as confirmed by OIS rates being lower than Libor, since in times of stress, the Libor, referencing a cash instrument, reflects both credit and liquidity risk. It is therefore becoming frequent among financial players to use the OIS swap curve to build a riskless term structure.

Historically, the spread has been around 10bp (for the period $01/2006 - 07/2007$), however in just over a month, the spread rose to 62 bp on 14/09/2007, when the Bank of England announced emergency funding to rescue the troubled Northern Rock, one of the UK's largest

mortgage lenders. The spread reached its all time high at 82 bp on 14/12/2007. Around the same time, large investment banks such as UBS and Lehman Brothers announced huge write downs. On 17/03/2008 the collapse of Bear Sterns led to a 74 bp spread. In the latest illiquidity wave following the failure of Lehman Brothers, the spread was 222 bp (as of 10/10/2008).

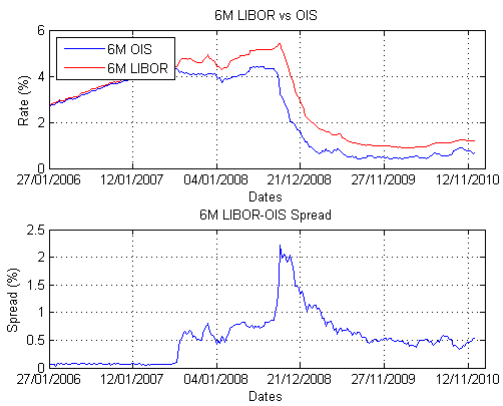


**Figure 1:** 6M LIBOR-OIS Spread

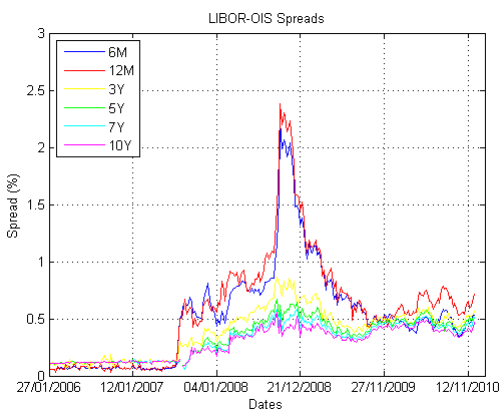


**Figure 2:** LIBOR-OIS Spreads

This discrepancy of values immediately raised issues in the construction of zero-coupon curves, which clearly, could no longer be based on traditional bootstrapping procedures. The assumption of a unique yield curve has been forsaken by practitioners, who seem to agree on an empirical approach based on the construction of as many curves as possible rate tenors. For each given contract, they select a specific discount curve, which they use to calculate the

net present value (NPV) of the contract's future payments, consistently with the contract's features and the counterparty in question. They then build as many forward LIBOR curves as given market tenors. With this approach, future cash flows are generated by the curves associated with the underlying rate tenors and their NPV is calculated through the selected discount curve.

The valuation of interest rate derivatives under different curves, for generating future rates and for discounting has received a lot of attention in the financial literature recently. Previous works that started to deal with these issues, mainly concerning the valuation of cross currency swaps, are Boenkost and Schmidt (2005), Kijima et.al.(2009) and Henrard (2009). Bianchetti (2009), was the first to apply the methodology to the single currency case, while Chibane and Sheldon (2009) propose methods to extend yield curve bootstrapping to a multi-curve setting. In terms of new pricing models, Kijima et.al. (2009) and Mercurio (2009,2010) apply the method using a stochastic volatility LMM, in order to price calets and swaptions, while Pallavicini and Tarenghi (2010), apply the new framework under the HJM model.

We follow the approach proposed by Mercurio (2010), to show how it can be used in practice when pricing other interest rate products starting from the constant maturity swap (CMS). CMS is a common money market product whose valuation is challenging. A standard approach to their valuation involves the calculation of a convexity adjustment. Unfortunately the convexity adjustment cannot be computed exactly and a number of assumptions and approximations have to be made. The chief of these are that the term structure of interest rates is flat, and may deform only through parallel shifts.

Following Hagan (2003), we apply the old convexity adjustment under the new framework of double curving. We also develop a new convexity adjustment, by assuming that term structure is not flat and may tilt. The new convexity adjustment is significantly larger than the old one and gives better fit to the market's CMS spread price.

## 2 Definitions and Notation

We assume, we are using two curves, one for the discounting, say curve "$D$", and one for the forwarding, say curve "$F$".

Forward rates can be defined for both curves. For each curve, the (simply compounded) forward rate prevailing at time $t$ and applied to the future time interval $[T, S]$, with $t \leq T < S$, is defined by:

$$F(t; T, S) = \frac{1}{\tau(T, S)} \left[ \frac{B(t, T)}{B(t, S)} - 1 \right] \tag{1}$$

where $\tau(T, S)$ is the year fraction for the interval $[T, S]$, and $B(t, T)$ and $B(t, S)$ are the discount factors (equivalently, zero coupon bond prices) at time $t$ for maturities $T$ and $S$.

We assume we are given a single discount curve to be used in the calculation of all NPVs, i.e. for discounting all future cash flows. This curve is assumed to be the OIS zero-coupon curve, stripped from market OIS swap rates and is defined for every possible maturity $T$: $T \to B_D(0, T) = B^{OIS}(0, T)$ where we denote by $B_D(t, T)$ the curve "$D$" discount factor at time $t$ for maturity $T$. All pricing measures we will consider are those associated to the OIS discount curve "$D$".

### 2.1 Definition of FRA and its properties

**Definition 2.1.** *Consider times $t$, $T_1$, $T_2$, with $t \leq T_1 < T_2$. The time-$t$ FRA rate $FRA(t; T_1, T_2)$ is defined as the fixed rate to be exchanged at time $T_2$ for the LIBOR rate $L(T_1, T_2)$, so that the swap has zero value at time $t$.*

We denote by $\mathbb{Q}_D^T$, the T-forward measure with numeraire the zero coupon bond $B_D(t, T)$. So, by (risk-adjusted) no-arbitrage pricing, we immediately have:

$$FRA(t; T_1, T_2) = E_D^{T_2} \left[ L(T_1, T_2) | \mathcal{F}_t \right] \tag{2}$$

where $E_D^{T_2}$ denotes expectation under the $T_2$-forward measure $\mathbb{Q}_D^{T_2}$ and $\mathcal{F}_t$ denotes the 'infor-

mation' available in the market at time $t$.

**Proposition 1.** *Any simple compounded forward rate spanning a time interval ending in $T$, is a martingale under the $T$-forward measure. i.e.*

$$F(u;S,T) = E^T\left[F(t;S,T)|\mathcal{F}_u\right] \tag{3}$$

*for:* $0 \leq u \leq t \leq S < T$

In the classic single-curve valuation, i.e. when the LIBOR curve corresponding to tenor $(T_2 - T_1)$ coincides with the discount curve (i.e. curves "$F$" and "$D$" are the same), the FRA rate $FRA(t;T_1,T_2)$ coincides with the forward LIBOR rate.

$$F_X(t;T_1,T_2) = \frac{1}{T_2 - T_1}\left[\frac{B_D(t,T_1)}{B_D(t,T_2)} - 1\right] = E_D^{T_2}\left[L(T_1,T_2)|\mathcal{F}_t\right] \tag{4}$$

In fact, the spot LIBOR rate $L(T_1,T_2)$ can be defined by the classic relation:

$$L(T_1,T_2) = \frac{1}{T_2 - T_1}\left[\frac{1}{B_D(T_1,T_2)} - 1\right] = F_X(T_1;T_1,T_2) \tag{5}$$

and FRA rate can be written as:

$$FRA(t;T_1,T_2) = E_D^{T_2}\left[F_X(T_1;T_1,T_2)|\mathcal{F}_t\right] \tag{6}$$

But by proposition 2.1, $F_X(t;T_1,T_2)$ is a martingale under $\mathbb{Q}_D^{T_2}$, i.e. $F_X(t;T_1,T_2) = E_D^{T_2}\left[L(T_1,T_2)|\mathcal{F}_t\right]$, and we conclude that: $FRA(t;T_1,T_2) = F_X(t;T_1,T_2)$.

In our dual-curve setting, however, *eq.*5 does not hold anymore, since the simply compounded rates defined by the discount curve are different, in general, from the corresponding LIBOR fixings.

The forward rate $F_X(t;T_1,T_2)$ is NOT a martingale under the forward measure $\mathbb{Q}_D^{T_2}$, and the FRA rate $FRA(t;T_1,T_2) \neq F_X(t;T_1,T_2)$.

Therefore, the present value of a future LIBOR rate is no longer obtained by discounting

the corresponding forward rate, but by discounting the corresponding FRA rate.

According to Mercurio (2010), the FRA rate is the natural generalization of a forward rate to the dual-curve case.

## 2.2 Definition of IRS

We show how to value an interest rate swap under the assumption of distinct forward and discount curves. For simplicity, we assume that IRS tenors for fixed and floating legs are the same (e.g. $6m$).

Let us consider a set of reset dates $\{T_i\}_{i=0,\ldots,n}$, with $T_i < T_{i+1}$ and set $\delta_i = T_{i+1} - T_i$.

The time-$t$ value ($t \leq T_0$) of the floating leg payoff can be obtained by taking the discounted expectation under the forward measure $\mathbb{Q}_D^{T_{i+1}}$:

$$FloatingLeg = \delta_i B_D(t, T_{i+1}) E_D^{T_{i+1}} \left[ L(T_i, T_{i+1}) | \mathcal{F}_t \right] \tag{7}$$

Defining the time-$t$ FRA rate as the fixed rate to be exchanged at time $T_{i+1}$ for the floating payment, so that the swap has zero value at time $t$. i.e.

$$FRA(t; T_i, T_{i+1}) = E_D^{T_{i+1}} \left[ L(T_i, T_{i+1}) | \mathcal{F}_t \right] \tag{8}$$

So, the floating leg is written as:

$$FloatingLeg = \delta_i B_D(t, T_{i+1}) FRA(t; T_i, T_{i+1}) \tag{9}$$

Therefore, the present value of the swap's floating leg is simply given by summing the values of simple payments, i.e:

$$FloatingLeg = \sum_{i=0}^{n-1} \delta_i B_D(t, T_{i+1}) FRA(t; T_i, T_{i+1}) \tag{10}$$

Let's consider now the swap's fixed leg and denote by $K$ the fixed rate payment on the fixed legs' dates $\{T_i\}_{i=1,\dots,n}$ (since we have assumed that both legs have the same tenors).

The present value of these payments is obtained by discounting them with the discount curve "$D$":

$$FixedLeg = \sum_{i=0}^{n-1} \delta_i K B_D(t, T_{i+1}) = K \sum_{i=0}^{n-1} \delta_i B_D(t, T_{i+1}) \tag{11}$$

Therefore the IRS value to the fixed rate payer is given by:

$$IRS(t, K; T_i) = \sum_{i=0}^{n-1} \delta_i B_D(t, T_{i+1}) FRA(t; T_i, T_{i+1}) - K \sum_{i=0}^{n-1} \delta_i B_D(t, T_{i+1}) \tag{12}$$

We can now calculate the corresponding forward swap rate as the fixed rate $K$ that makes the IRS value equal to zero at time $t$:

$$S_i(t) = \frac{\sum_{i=0}^{n-1} \delta_i B_D(t, T_{i+1}) FRA(t; T_i, T_{i+1})}{\sum_{i=0}^{n-1} \delta_i B_D(t, T_{i+1})} \tag{13}$$

This is the forward swap rate of an IRS where cash flows are generated through curve "$F$" and discounted with curve "$D$".

### 2.3 Definition of Constant Maturity Swap

A constant maturity swap (CMS) contract exchanges a swap rate with a certain time to maturity $c$, against a fixed payment $K$. In a standard swap, we exchange a LIBOR against a fixed payment. In the CMS, the floating rate is no longer a LIBOR, but a swap rate with a certain time to maturity, denoted by $c$.

For $t \leq T$, let $B_D(t, T)$ denote the value at time $t$ of the curve "$D$" discount factor.

Let $\mathcal{T} = \{T_i\}_{i=0,\dots,n}$, with $T_i < T_{i+1}$, be a set of reset dates and set $\delta_i = T_{i+1} - T_i$. Write $B_t^{T_i}$ for $B_D(t, T_i)$. We also write $S_t^{\mathcal{T}}$ for the value at time $t \leq T_0$ of the forward swap rate

corresponding to the reset dates $\mathcal{T}$:

$$S_t^{\mathcal{T}} = \frac{\sum_{i=0}^{n-1} \delta_i B_t^{T_{i+1}} FRA(t; T_i, T_{i+1})}{\sum_{i=0}^{n-1} \delta_i B_t^{T_{i+1}}} \tag{14}$$

We shall assume that $\delta_i = \delta$ is constant (i.e. $3m$).

Let $t_i = \{t_{i,j}\}_{j=0,\ldots,c}$, be a set of associated reset dates, associated with the date $T_i$, $i = 0, \ldots, n$, with $t_{i,0} \in [T_i, T_{i+1}]$ and $t_{i,j} - t_{i,j-1} = \tau$, with $j = 1, \ldots, c$. (we suppose $\tau = 6m$)

We will suppose for simplicity that $t_{i,0} = T_i$, and we will set $\Delta = \delta/\tau$, with $\Delta$ need not be integral.

The forward swap rate of the $i^{th}$ IRS at time determined by $t_i$, at time $t \leq t_{i,j-1}$ is $S_t^{t_i} \equiv S_t^i$:

$$S_t^i = \frac{\sum_{j=1}^{c} \tau B_t(t_{i,j}) FRA(t; t_{i,j-1}, t_{i,j})}{\sum_{j=1}^{c} \tau B_t(t_{i,j})} \tag{15}$$

where $B_t(t_{i,j}) = B_D(t, t_{i,j})$

Write $t = \{t_i\}_{i=0,\ldots,n}$ for the set of associated reset dates. $t$ defines a CMS as follows: At times $T_i$, $i = 1, \ldots, n$, the two counterparties to the CMS, $A$ and $B$, exchange cashflows:

$$\text{A pays B: } \delta L_{T_{i-1}}^{i-1} + \delta X^t$$

$$\text{B pays A: } \delta S_{T_{i-1}}^{i-1}$$

where:

$L_{T_{i-1}^{i-1}}$, is the $\delta$ spot LIBOR rate applied to a $\delta$ period $[T_{i-1}, T_i]$

$S_{T_{i-1}^{i-1}}$, is the $\tau$ spot Swap rate applied to a $\delta$ period $[T_{i-1}, T_i]$

$X^t$, is the CMS premium (spread), a constant chosen so that the cost of the instrument at time $t$, when the contract is initiated, is zero.

$X^t$ cannot be computed exactly. Usually an approximation is used, computed via a con-

vexity adjustment, or a numerical method such as Monte Carlo.

The cashflow $p_i$ at time $T_i$ to a CMS is:

$$p_i = \delta \left[ S^{i-1}_{T_{i-1}} - L^{i-1}_{T_{i-1}} - X^t \right] \tag{16}$$

where we suppose that the counterparty (say $B$) pays floating (i.e. LIBOR + Spread) and receives fixed (i.e. the swap rate).

The time-$t$ value, with $t \leq T_{i-1}$, of the CMS can be obtained by taking the discounted expectation $E^{T_i}_D$ under the forward measure $\mathbb{Q}^{T_i}_D$ corresponding to the numeraire $B^{T_i}_t = B_D(t, T_i)$. So:

$$v^{T_i}_t = B^{T_i}_t \delta E^{T_i}_D \left[ \frac{S^{i-1}_{T_{i-1}} - L^{i-1}_{T_{i-1}} - X^t}{B^{T_i}_{T_i}} \right] = B^{T_i}_t \delta E^{T_i}_D \left[ S^{i-1}_{T_{i-1}} - L^{i-1}_{T_{i-1}} - X^t \right] \tag{17}$$

But $E^{T_i}_D \left[ X^t \right] = X^t$, since $X^t$ is a constant, and $E^{T_i}_D \left[ L^{i-1}_{T_{i-1}} \right] = FRA(t; T_{i-1}, T_i)$ is a martingale under the forward measure $\mathbb{Q}^{T_i}_D$.

So, the time-$t$ value of the CMS is given by:

$$v^{T_i}_t = B^{T_i}_t \delta \left( E^{T_i}_D \left[ S^{i-1}_{T_{i-1}} \right] - FRA(t; T_{i-1}, T_i) - X^t \right) \tag{18}$$

A convexity adjustment is applied to the term $E^{T_i}_D \left[ S^{i-1}_{T_{i-1}} \right]$.

Write $\mathbb{Q}^{t_i}_D$ for the forward swap measure corresponding to the numeraire $B^{t_i}_t = \sum_{j=1}^{c} \tau B_t(t_{i,j})$ (i.e. the denominator of the forward swap rate), under which, forward swap rate $S^i_t$ is a martingale.

## 3 The Convexity Adjustment

Let $M$ and $N$ be numeraires, with corresponding martingale measures $\mathbb{M}$ and $\mathbb{N}$. Suppose that a process $X$ is a martingale under $\mathbb{M}$ and that we want to calculate:

$$E_t^{\mathbb{N}}[X_T] = E^{\mathbb{N}}[X_T|\mathcal{F}_t] \tag{19}$$

its expected future value under the non-martingale measure. Trivially,

$$E_t^{\mathbb{N}}[X_T] = E_t^{\mathbb{M}}[X_T] + CA_t \tag{20}$$

where the convexity adjustment $CA_t$ is

$$CA_t = E_t^{\mathbb{N}}[X_T] - E_t^{\mathbb{M}}[X_T] \tag{21}$$

The convexity adjustment is simply the difference in expectation of some quantity when the expectations are computed under two different measures (see Pelsser (2003)).

Changing numeraire in the first term

$$CA_t = E_t^{\mathbb{M}}\left[X_T \frac{N_T/N_t}{M_T/M_t}\right] - E_t^{\mathbb{M}}[X_T] = E_t^{\mathbb{M}}\left[X_T\left(\frac{N_T/N_t}{M_T/M_t} - 1\right)\right] \tag{22}$$

We apply this in our case. So, the expectation $E_D^{T_i}\left[S_{T_{i-1}}^{i-1}\right]$ we want to calculate can be written as an expectation which is a martingale under its measure plus an adjustment. So,

$$E_D^{T_i}\left[S_{T_{i-1}}^{i-1}\right] = E_D^{t_{i-1}}\left[S_{T_{i-1}}^{i-1}\right] + CA_t^{t_{i-1}} \tag{23}$$

where, under the forward swap measure $\mathbb{Q}_D^{t_i}$, the forward swap rate is a martingale. So, the convexity adjustment is simply the difference in expectation of some quantity when the

expectations are computed under two different measures. i.e.

$$CA_t^{t_{i-1}} = E_D^{T_i}\left[S_{T_{i-1}}^{i-1}\right] - E_D^{t_{i-1}}\left[S_{T_{i-1}}^{i-1}\right] \tag{24}$$

Suppose we have found the convexity adjustment $CA_t^{t_{i-1}}$. Then, the current value of the CMS is given by:

$$v_t^{T_i} = B_t^{T_i}\delta\left(E_D^{T_i}\left[S_{T_{i-1}}^{i-1}\right] - FRA(t;T_{i-1},T_i) - X^t\right) \tag{25}$$

$$= B_t^{T_i}\delta\left(E_D^{t_{i-1}}\left[S_{T_{i-1}}^{i-1}\right] + CA_t^{t_{i-1}} - FRA(t;T_{i-1},T_i) - X^t\right) \tag{26}$$

But $S_{T_{i-1}}^{i-1}$ is a martingale under the forward swap measure $\mathbb{Q}_D^{t_i}$, which means that $E_D^{t_{i-1}}\left[S_{T_{i-1}}^{i-1}\right] = S_t^{i-1}$. So, the current value is given by:

$$v_t^{T_i} = B_t^{T_i}\delta\left[S_t^{i-1} + CA_t^{t_{i-1}} - FRA(t;T_{i-1},T_i) - X^t\right] \tag{27}$$

and the present value of the CMS is given by:

$$v_t = \sum_{i=1}^{n} v_t^{T_i} = \sum_{i=1}^{n} B_t^{T_i}\delta\left[S_t^{i-1} + CA_t^{t_{i-1}} - FRA(t;T_{i-1},T_i) - X^t\right] \tag{28}$$

$$= \sum_{i=1}^{n} B_t^{T_i}\delta\left[S_t^{i-1} + CA_t^{t_{i-1}}\right] - \sum_{i=1}^{n}\delta B_t^{T_i} FRA(t;T_{i-1},T_i) - \sum_{i=1}^{n} B_t^{T_i}\delta X^t \tag{29}$$

Since $X^t$ is determined by the requirement that the CMS swap is a zero cost instrument, set $v_t = 0$, and so we have:

$$X^t = \frac{\sum_{i=1}^{n} B_t^{T_i}\delta\left[S_t^{i-1} + CA_t^{t_{i-1}}\right] - \sum_{i=1}^{n}\delta B_t^{T_i} FRA(t;T_{i-1},T_i)}{\sum_{i=1}^{n}\delta B_t^{T_i}} \tag{30}$$

$$= \frac{\sum_{i=1}^{n} B_t^{T_i}\delta\left[S_t^{i-1} + CA_t^{t_{i-1}}\right]}{\sum_{i=1}^{n}\delta B_t^{T_i}} - \frac{\sum_{i=1}^{n}\delta B_t^{T_i} FRA(t;T_{i-1},T_i)}{\sum_{i=1}^{n}\delta B_t^{T_i}} \tag{31}$$

$$= \frac{\sum_{i=1}^{n} B_t^{T_i} \delta \left[ S_t^{i-1} + CA_t^{t_{i-1}} \right]}{\sum_{i=1}^{n} \delta B_t^{T_i}} - S_t^{\mathcal{T}} \tag{32}$$

The sole problem is thus to determine the convexity adjustment $CA_t^{t_{i-1}}$.

From equations 22 and 24, we have:

$$CA_t^{t_{i-1}} = E_D^{t_{i-1}} \left[ S_{T_{i-1}}^{i-1} \left( \frac{B_{T_{i-1}}^{T_i} / B_t^{T_i}}{B_{T_{i-1}}^{t_{i-1}} / B_t^{t_{i-1}}} - 1 \right) \right] \tag{33}$$

$$= E_D^{t_{i-1}} \left[ S_{T_{i-1}}^{i-1} \left( \frac{B_{T_{i-1}}^{T_i}}{B_t^{T_i}} \frac{B_t^{t_{i-1}}}{B_{T_{i-1}}^{t_{i-1}}} - 1 \right) \right] \tag{34}$$

where $\frac{B_t^{t_{i-1}}}{B_{T_{i-1}}^{t_{i-1}}}$ are the numeraires of the forward swap measure $Q_D^{t_i}$.

For $i \in \{1, \ldots, n\}$ define

$$G_t^i = \frac{B_t^{T_i}}{B_t^{t_{i-1}}} \tag{35}$$

for $0 \leq t \leq T_{i-1}$. Then we have,

$$CA_t^{t_{i-1}} = E_D^{t_{i-1}} \left[ S_{T_{i-1}}^{i-1} \left( \frac{G_{T_{i-1}}^i}{G_t^i} - 1 \right) \right] \tag{36}$$

We approximate $CA_t^{t_{i-1}}$ by finding an approximation to $G_t^i$.

We start by first deriving a well known expression for the convexity adjustment when the term structure is flat and has only parallel shifts.

### 3.1 Flat term structure with parallel shifts

Suppose that the term structure is flat and has only parallel shifts. We represent its level using a rate of tenor $\tau$. Write $r_t$ for the value at time $t$ of the tenor $\tau$ rate so that for $t \leq T$

$$B_t^T = (1 + \tau r_t)^{-(T-t)/\tau} \tag{37}$$

For $t \leq T_{i-1}$ we have:

$$\left.\begin{aligned} B_t^{T_i} &= (1+\tau r_t)^{-\frac{(T_i-t)}{\tau}} \\ B_t^{T_{i-1}} &= (1+\tau r_t)^{-\frac{(T_{i-1}-t)}{\tau}} \end{aligned}\right\} \Rightarrow \frac{B_t^{T_i}}{B_t^{T_{i-1}}} = (1+\tau r_t)^{-\frac{(T_i-T_{i-1})}{\tau}} = (1+\tau r_t)^{-\frac{\delta}{\tau}} \tag{38}$$

So,

$$B_t^{T_i} = \frac{1}{(1+\tau r_t)^{\Delta}} B_t^{T_{i-1}} \tag{39}$$

We also have:

$$\left.\begin{aligned} B_t^{t_{i,j}} &= (1+\tau r_t)^{-\frac{(t_{i,j}-t)}{\tau}} \\ B_t^{T_i} &= (1+\tau r_t)^{-\frac{(T_i-t)}{\tau}} \end{aligned}\right\} \Rightarrow \frac{B_t^{t_{i,j}}}{B_t^{T_i}} = (1+\tau r_t)^{-\frac{(t_{i,j}-T_i)}{\tau}} = (1+\tau r_t)^{-j} \tag{40}$$

So,

$$B_t^{t_{i,j}} = \frac{1}{(1+\tau r_t)^{j}} B_t^{T_i} \tag{41}$$

for $j = 0, \ldots, c$

Then, for $t \leq T_{i-1}$:

$$B_t^{t_i} = \sum_{j=1}^{c} \tau B_t^{t_{i,j}} = \sum_{j=1}^{c} \tau \frac{1}{(1+\tau r_t)^{j}} B_t^{T_i} \tag{42}$$

But:

$$\sum_{j=1}^{c} \tau \frac{1}{(1+\tau r_t)^{j}} = \frac{\tau}{1+\tau r_t}\left[1 + \left(\frac{1}{1+\tau r_t}\right) + \left(\frac{1}{1+\tau r_t}\right)^2 + \ldots + \left(\frac{1}{1+\tau r_t}^2\right)^{c-1}\right] \tag{43}$$

$$= \frac{\tau}{1+\tau r_t}\left[\frac{1-\left(\frac{1}{1+\tau r_t}\right)^c}{1-\left(\frac{1}{1+\tau r_t}\right)}\right] = \frac{1}{r_t}\left[1 - \frac{1}{(1+\tau r_t)^c}\right] \tag{44}$$

So,

$$B_t^{t_i} = B_t^{T_i} \frac{1}{r_t}\left[1 - \frac{1}{(1+\tau r_t)^c}\right] \tag{45}$$

The forward swap rate is written as:

$$S_t^i = \frac{\sum_{j=1}^{c} \tau B_t(t_{i,j}) FRA(t; t_{i,j-1}, t_{i,j})}{\sum_{j=1}^{c} \tau B_t(t_{i,j})} \tag{46}$$

But, since we suppose to have a flat term structure, the forward rate $FRA(t; t_{i,j-1}, t_{i,j})$, does not depend on $j$ (i.e. constant), and so goes out of the summation. So, under this assumption, the forward swap rate is given by:

$$S_t^i = \sum_{j=1}^{c} W_j FRA(t; t_{i,j-1}, t_{i,j}) = R_t \tag{47}$$

with:

$$W_j = \frac{B_t(t_{i,j})}{\sum_{j=1}^{c} B_t(t_{i,j})} \tag{48}$$

where $R_t$ is supposed to be the risky rate.

So, since $R_t$ is a risky rate, following Liu et.al. (2006), we can assume that is equal to a risk-free rate $r_t$ (e.g. OIS rate) plus a spread $X_t$. So, we can assume that the forward swap rate is written as:

$$S_t^i = R_t = r_t + X_t \tag{49}$$

where the spread $X_t$, represents the credit or liquidity risk of the counterparty.

### 3.1.1 The first approximation

Putting together equations 39 and 45, we get an expression for $G_t^i$ that depends on $t$ only via $r_t$. In fact, $G_t^i = G(r_t)$ where:

$$G(r) = \frac{r}{(1+\tau r)^{\Delta}} \frac{1}{1 - \frac{1}{(1+\tau r)^c}} \tag{50}$$

So, $G_t^i$ is given by:

$$G_t^i = \frac{B_t^{T_i}}{B_t^{t_{i-1}}} = \frac{B_t^{T_{i-1}} (1+\tau r)^{-\Delta}}{B_t^{T_{i-1}} \frac{1}{r_t} \left[1 - \frac{1}{(1+\tau r_t)^c}\right]} = \frac{r_t}{(1+\tau r_t)^{\Delta}} \frac{1}{1 - \frac{1}{(1+\tau r_t)^c}} = G(r_t) \tag{51}$$

So far no approximation has been made. To progress, a Taylor series expansion is made on $G$. So, a first order Taylor expansion of $G$ at $r_t$ gives:

$$G(r) \approx G(r_t) + G'(r_t)(r - r_t) \Rightarrow \frac{G(r)}{G(r_t)} - 1 = \frac{G'(r_t)}{G(r_t)}(r - r_t) \tag{52}$$

Assuming that $r_{T_{i-1}}$ is a perturbation of $r_t$, and since:

$$\frac{G^i_{T_{i-1}}}{G^i_t} - 1 = \frac{G(r_{T_{i-1}})}{G(r_t)} - 1 \approx \frac{G'(r_t)}{G(r_t)}(r_{T_{i-1}} - r_t) \tag{53}$$

$$= \frac{G'\left(S^{i-1}_t - X_t\right)}{G(S^{i-1}_t - X_t)}\left[S^{i-1}_{T_{i-1}} - X_{T_{i-1}} - S^{i-1}_t + X_t\right] \tag{54}$$

So, the convexity adjustment can now be approximated as:

$$CA^{t_{i-1}}_t = E^{t_{i-1}}_D\left[S^{i-1}_{T_{i-1}}\left(\frac{G^i_{T_{i-1}}}{G^i_t} - 1\right)\right] \tag{55}$$

$$= E^{t_{i-1}}_D\left[S^{i-1}_{T_{i-1}}\frac{G'\left(S^{i-1}_t - X_t\right)}{G(S^{i-1}_t - X_t)}\left[S^{i-1}_{T_{i-1}} - X_{T_{i-1}} - S^{i-1}_t + X_t\right]\right] \tag{56}$$

$$= \frac{G'\left(S^{i-1}_t - X_t\right)}{G(S^{i-1}_t - X_t)}E^{t_{i-1}}_D\left[\left(S^{i-1}_{T_{i-1}}\right)^2 - S^{i-1}_{T_{i-1}}\left(S^{i-1}_t + X_{T_{i-1}} - X_t\right)\right] \tag{57}$$

$$= \frac{G'\left(S^{i-1}_t - X_t\right)}{G(S^{i-1}_t - X_t)}\left[E^{t_{i-1}}_D\left[\left(S^{i-1}_{T_{i-1}}\right)^2\right] - E^{t_{i-1}}_D\left[S^{i-1}_{T_{i-1}}X_{T_{i-1}}\right] - \left(S^{i-1}_t\right)^2 + \left(S^{i-1}_t X_t\right)\right] \tag{58}$$

$$= \left(S^{i-1}_t\right)^2\frac{G'\left(S^{i-1}_t - X_t\right)}{G(S^{i-1}_t - X_t)}\left[\left(E^{t_{i-1}}_D\left[\left(\frac{S^{i-1}_{T_{i-1}}}{S^{i-1}_t}\right)^2\right] - 1\right) - \left(E^{t_{i-1}}_D\left[\frac{S^{i-1}_{T_{i-1}}X_{T_{i-1}}}{\left(S^{i-1}_t\right)^2}\right] - \frac{X_t}{S^{i-1}_t}\right)\right] \tag{59}$$

It is easy to find an expression for $\frac{G'\left(S^{i-1}_t - X_t\right)}{G(S^{i-1}_t - X_t)}$. In fact:

$$\frac{G'(r)}{G(r)} = \frac{1}{r}\theta(r) \tag{60}$$

where:

$$\theta(r) = \frac{1}{1 + \tau r}\left(1 + (\tau - \delta)r - \frac{c\tau r}{(1 + \tau r)^c - 1}\right) \tag{61}$$

So, replacing, the convexity adjustment is given by:

$$CA_t^{t_{i-1}} \approx \frac{\left(S_t^{i-1}\right)^2}{S_t^{i-1} - X_t}\theta\left(S_t^{i-1} - X_t\right)\left[\left(E_D^{t_{i-1}}\left[\left(\frac{S_{T_{i-1}}^{i-1}}{S_t^{i-1}}\right)^2\right] - 1\right) - \left(E_D^{t_{i-1}}\left[\frac{S_{T_{i-1}}^{i-1}X_{T_{i-1}}}{\left(S_t^{i-1}\right)^2}\right] - \frac{X_t}{S_t^{i-1}}\right)\right] \tag{62}$$

### 3.1.2 The second approximation

We need to calculate $E_D^{t_{i-1}}\left[\left(S_{T_{i-1}}^{i-1}\right)^2\right]$ and $E_D^{t_{i-1}}\left[S_{T_{i-1}}^{i-1}X_{T_{i-1}}\right]$. We assume that under the measure $Q_D^{t_{i-1}}$ the two processes (i.e. swap rate and spread), are martingales and are of the form:

$$dS_t^{i-1} = \sigma_{t,s}^{i-1}S_t^{i-1}dW_{t,s}^{i-1} \tag{63}$$

$$dX_t = \sigma_{t,x}X_t dW_{t,x} \tag{64}$$

where $W_{t,s}^{i-1}$ and $W_{t,x}$ are two correlated wiener processes with correlation $\rho^{i-1}$ and $\sigma_{t,s}^{i-1}$ and $\sigma_{t,x}$ are deterministic volatilities.

Applying Ito's Lemma, the two expectations are given as:

$$E_D^{t_{i-1}}\left[\left(S_{T_{i-1}}^{i-1}\right)^2\right] = \left(S_t^{i-1}\right)^2\exp\left(\left(\sigma_{t,s}^{i-1}\right)^2\left(T_{i-1} - t\right)\right) \tag{65}$$

$$E_D^{t_{i-1}}\left[S_{T_{i-1}}^{i-1}X_{T_{i-1}}\right] = S_t^{i-1}X_t\exp\left(\rho^{i-1}\sigma_{t,s}^{i-1}\sigma_{t,x}\left(T_{i-1} - t\right)\right) \tag{66}$$

### 3.1.3 The convexity adjustment

Putting everything together, we arrive at the following expression for the convexity adjustment:

$$CA_t^{t_{i-1}} \approx \frac{\left(S_t^{i-1}\right)^2}{S_t^{i-1} - X_t}\theta\left(S_t^{i-1} - X_t\right)\left[\left(e^{\left(\sigma_{t,s}^{i-1}\right)^2\left(T_{i-1}-t\right)} - 1\right) - \frac{X_t}{S_t^{i-1}}\left(e^{\rho^{i-1}\sigma_{t,s}^{i-1}\sigma_{t,x}\left(T_{i-1}-t\right)} - 1\right)\right] \tag{67}$$

Note that $\left(S_t^{i-1} - X_t\right)\theta\left(S_t^{i-1} - X_t\right) = r_t\theta(r_t)$ does not depend on $i$. Hence, the final expression for $CA_t^{t_{i-1}}$, with the same form for all $i$, is:

$$CA_t^{t_{i-1}} = K(r_t)\left[\left(e^{\left(\sigma_{t,s}^{i-1}\right)^2(T_{i-1}-t)} - 1\right) - \frac{X_t}{S_t^{i-1}}\left(e^{\rho^{i-1}\sigma_{t,s}^{i-1}\sigma_{t,x}(T_{i-1}-t)} - 1\right)\right] \tag{68}$$

with

$$K(r_t) = \frac{(S_t^{i-1})^2}{r_t}\frac{1}{1+\tau r_t}\left(1 + (\tau-\delta)r_t - \frac{c\tau r_t}{(1+\tau r_t)^c - 1}\right) \tag{69}$$

As we can easily understand, by assuming that the there is no spread in the market, i.e. $X_t = 0$, we end up with the well-known formula proposed by Hagan (2003).

### 3.2 A term structure with tilts

We introduce a new convexity adjustment by extending the analysis of the previous section to the case when the term structure may tilt.

For $t \leq T$ we write:

$$B_t^T = \left(1 + \tau r_t^T\right)^{-(T-t)/\tau} \tag{70}$$

where $r_t^T$ is the spot rate at time $t$ for $B_t^T$ compounded with tenor $\tau$. We no longer assume the term structure is flat. Instead, we assume it is given by some deterministic function $f$. Write:

$$r_t^T = f(r_t, t, T \mid a) \tag{71}$$

where $r_t$ is the short rate and $a = (a_1, \ldots, a_k)$ is some vector of parameters.

For $i \in \{1, \ldots, n\}$, we define as before:

$$G_t^i = \frac{B_t^{T_i}}{B_t^{t_{i-1}}} \tag{72}$$

for $0 \leq t \leq T_{i-1}$. Then we have,

$$CA_t^{t_{i-1}} = E_D^{t_{i-1}} \left[ S_{T_{i-1}}^{i-1} \left( \frac{G_{T_{i-1}}^i}{G_t^i} - 1 \right) \right] \tag{73}$$

But from equations (42) and (69) we have that:

$$\left. \begin{array}{l} B_t^{T_i} = \left(1 + \tau r_t^{T_i}\right)^{-\frac{(T_i - t)}{\tau}} \\ B_t^{t_{i-1}} = \sum_{j=1}^c \tau B_t^{t_{i,j}} = \sum_{j=1}^c \tau \left(1 + \tau r_t^{t_{i,j}}\right)^{-\frac{(t_{i,j} - t)}{\tau}} \end{array} \right\} \Rightarrow \tag{74}$$

$$\Rightarrow G_t^i = \frac{B_t^{T_i}}{B_t^{t_{i-1}}} = \frac{\left(1 + \tau r_t^{T_i}\right)^{-\frac{(T_i - t)}{\tau}}}{\sum_{j=1}^c \tau \left(1 + \tau r_t^{t_{i,j}}\right)^{-\frac{(t_{i,j} - t)}{\tau}}} = \frac{\left(1 + \tau f_t^i(r_t)\right)^{-\frac{(T_i - t)}{\tau}}}{\sum_{j=1}^c \tau \left(1 + \tau f_t^{i,j}(r_t)\right)^{-\frac{(t_{i,j} - t)}{\tau}}} \tag{75}$$

where we have written $f_t^i(r_t)$ for $r_t^{T_i} = f(r_t, t, T_i)$ and $f_t^{i,j}(r_t)$ for $r_t^{t_{i,j}} = f(r_t, t, t_{i,j})$.

Now fix $i$ and for $t \leq T_{i-1}$ define:

$$G(r, t) = \frac{\left(1 + \tau f_t^i(r)\right)^{-\frac{(T_i - t)}{\tau}}}{\sum_{j=1}^c \tau \left(1 + \tau f_t^{i,j}(r)\right)^{-\frac{(t_{i,j} - t)}{\tau}}} \tag{76}$$

so that $G_t^i = G(r_t, t)$.

As before, we find an approximation of the convexity adjustment by using a Taylor expansion of $G(r, t)$.

A first order Taylor expansion of $G$ at $(r_t, t)$ gives:

$$G(r_{T_{i-1}}, T_{i-1}) \approx G(r_t, t) + G_r(r_t, t)(r_{T_{i-1}} - r_t) + G_t(r_t, t)(T_{i-1} - t) \tag{77}$$

where here $G_r$ and $G_t$ denote the partial derivatives of $G$ with respect to $r$ and $t$. Hence:

$$\frac{G_{T_{i-1}}^i}{G_t^i} - 1 = \frac{G(r_{T_{i-1}}, T_{i-1})}{G(r_t, t)} - 1 \approx \frac{G_r(r_t, t)}{G(r_t, t)}(r_{T_{i-1}} - r_t) + \frac{G_t(r_t, t)}{G(r_t, t)}(T_{i-1} - t) \tag{78}$$

In the previous case we had $S_t^i = R_t$. In the current case, where we are not assuming that the term structure is flat, we assume as approximations that $S_t^i \approx R_t$ and $S_{T_{i-1}}^i \approx R_{T_{i-1}}$. With these assumptions, we have:

$$S_{T_{i-1}}^{i-1}\left(\frac{G_{T_{i-1}}^i}{G_t^i}-1\right) \approx S_{T_{i-1}}^{i-1}\left[\frac{G_r(r_t,t)}{G(r_t,t)}(r_{T_{i-1}}-r_t)\right] + S_{T_{i-1}}^{i-1}\left[\frac{G_t(r_t,t)}{G(r_t,t)}(T_{i-1}-t)\right] \tag{79}$$

Substituting back into (72),

$$CA_t^{t_{i-1}} \approx E_D^{t_{i-1}}\left[S_{T_{i-1}}^{i-1}\frac{G_r(r_t,t)}{G(r_t,t)}(r_{T_{i-1}}-r_t)\right] + E_D^{t_{i-1}}\left[S_{T_{i-1}}^{i-1}\frac{G_t(r_t,t)}{G(r_t,t)}(T_{i-1}-t)\right] \tag{80}$$

$$= \frac{G_r(r_t,t)}{G(r_t,t)}E_D^{t_{i-1}}\left[S_{T_{i-1}}^{i-1}\left(S_{T_{i-1}}^{i-1}-X_{T_{i-1}}-S_t^{i-1}+X_t\right)\right] + \frac{G_t(r_t,t)}{G(r_t,t)}E_D^{t_{i-1}}\left[S_{T_{i-1}}^{i-1}(T_{i-1}-t)\right] \tag{81}$$

$$= \frac{G_r(r_t,t)}{G(r_t,t)}\left[E_D^{t_{i-1}}\left[\left(S_{T_{i-1}}^{i-1}\right)^2\right] - E_D^{t_{i-1}}\left[S_{T_{i-1}}^{i-1}X_{T_{i-1}}\right] - \left(S_t^{i-1}\right)^2 + \left(S_t^{i-1}X_t\right)\right] + \frac{G_t(r_t,t)}{G(r_t,t)}S_t^{i-1}(T_{i-1}-t) \tag{82}$$

$$= \left(S_t^{i-1}\right)^2\frac{G_r(r_t,t)}{G(r_t,t)}\left[\left(E_D^{t_{i-1}}\left[\left(\frac{S_{T_{i-1}}^{i-1}}{S_t^{i-1}}\right)^2\right]-1\right) - \left(E_D^{t_{i-1}}\left[\frac{S_{T_{i-1}}^{i-1}X_{T_{i-1}}}{\left(S_t^{i-1}\right)^2}\right] - \frac{X_t}{S_t^{i-1}}\right)\right] + \frac{G_t(r_t,t)}{G(r_t,t)}S_t^{i-1}(T_{i-1}-t) \tag{83}$$

We need to calculate $E_D^{t_{i-1}}\left[\left(S_{T_{i-1}}^{i-1}\right)^2\right]$ and $E_D^{t_{i-1}}\left[S_{T_{i-1}}^{i-1}X_{T_{i-1}}\right]$. As before, we assume that under the measure $Q_D^{t_{i-1}}$ the two processes (i.e. swap rate and spread), are martingales and are of the form:

$$dS_t^{i-1} = \sigma_{t,s}^{i-1}S_t^{i-1}dW_{t,s}^{i-1} \tag{84}$$

$$dX_t = \sigma_{t,x}X_t dW_{t,x} \tag{85}$$

where $W_{t,s}^{i-1}$ and $W_{t,x}$ are two correlated wiener processes with correlation $\rho^{i-1}$ and $\sigma_{t,s}^{i-1}$ and $\sigma_{t,x}$ are deterministic volatilities.

Applying Ito's Lemma, the two expectations are given as:

$$E_D^{t_{i-1}}\left[\left(S_{T_{i-1}}^{i-1}\right)^2\right] = \left(S_t^{i-1}\right)^2\exp\left(\left(\sigma_{t,s}^{i-1}\right)^2(T_{i-1}-t)\right) \tag{86}$$

$$E_D^{t_{i-1}}\left[S_{T_{i-1}}^{i-1}X_{T_{i-1}}\right] = S_t^{i-1}X_t\exp\left(\rho^{i-1}\sigma_{t,s}^{i-1}\sigma_{t,x}(T_{i-1}-t)\right) \tag{87}$$

So, the convexity adjustment is now given by:

$$\begin{aligned} CA_t^{t_{i-1}} &= \left(S_t^{i-1}\right)^2 \frac{G_r(r_t,t)}{G(r_t,t)} \left[\left(e^{\left(\sigma_{t,s}^{i-1}\right)^2 (T_{i-1}-t)} - 1\right) - \frac{X_t}{S_t^{i-1}} \left(e^{\rho^{i-1}\sigma_{t,s}^{i-1}\sigma_{t,x}(T_{i-1}-t)} - 1\right)\right] \\ &+ \frac{G_t(r_t,t)}{G(r_t,t)} S_t^{i-1}(T_{i-1}-t) \end{aligned} \tag{88}$$

Finally,we need to find expressions for the two partial derivatives $\frac{G_r(r,t)}{G(r,t)}$ and $\frac{G_t(r,t)}{G(r,t)}$. In fact, one can show that:

$$\frac{G_r(r,t)}{G(r,t)} = \frac{-(T_i-t)}{1+\tau f_t^i(r)} \frac{\partial f_t^i}{\partial r}(r) + \frac{\sum_{j=1}^c (t_{i,j}-t)(1+\tau f_t^{i,j}(r))^{-\frac{t_{i,j}-t}{\tau}-1} \frac{\partial f_t^{i,j}}{\partial r}(r)}{\sum_{j=1}^c (1+\tau f_t^{i,j}(r))^{-\frac{t_{i,j}-t}{\tau}}} \tag{89}$$

$$\begin{aligned} \frac{G_t(r,t)}{G(r,t)} &= \frac{1}{\tau}\ln(1+\tau f_t^i(r)) - \frac{T_i-t}{1+\tau f_t^i(r)} \frac{\partial f_t^i}{\partial r}(r) \\ &- \frac{\sum_{j=1}^c (1+\tau f_t^{i,j}(r))^{-\frac{t_{i,j}-t}{\tau}} \left(\frac{1}{\tau}\ln(1+\tau f_t^{i,j}(r)) - \frac{t_{i,j}-t}{1+\tau f_t^{i,j}(r)} \frac{\partial f_t^{i,j}}{\partial r}(r)\right)}{\sum_{j=1}^c (1+\tau f_t^{i,j}(r))^{-\frac{t_{i,j}-t}{\tau}}} \end{aligned} \tag{90}$$

To progress, we must specify $f$. We investigate the case:

$$f(r,t,T \mid a,b,k) = r + (a+b(T-t))e^{-k(T-t)} - a \tag{91}$$

based on the well-known Nelson and Siegel functional form. Then we have:

$$\frac{\partial f_t^i}{\partial r}(r) = \frac{\partial f_t^{i,j}}{\partial r}(r) = 1 \tag{92}$$

$$\frac{\partial f_t^i}{\partial t}(r) = (k(a+b(T_i-t))-b)e^{-k(T_i-t)} \tag{93}$$

$$\frac{\partial f_t^{i,j}}{\partial t}(r) = (k(a+b(t_{i,j}-t))-b)e^{-k(t_{i,j}-t)} \tag{94}$$

# 4 Smile-consistent Convexity Adjustment

In order to test our CMS convexity adjustments, we compare them with the smile-consistent convexity adjustment, widely used in the market.

In presence of market smile, when the term structure is not flat, but may tilt, the adjustment is necessarily more involved, if we aim to incorporate consistently the information coming from the quoted implied volatilities. The procedure to derive a smile consistent convexity adjustment is described in Mercurio and Pallavicini (2006),and is the one we will use here.

The expectations of the swap rates $E_D^{T_i}\left[S_{T_{i-1}}^{i-1}\right]$, are evaluated, under the discounting forward measure $Q_D^{T_i}$, by using the SABR model according to market practice.

The second moment of $S_{T_{i-1}}^{i-1}$ can be replicated exactly as follows:

$$E_D^{t_{i-1}}\left[\left(S_{T_{i-1}}^{i-1}\right)^2\right] = 2\int_0^\infty E_D^{t_{i-1}}\left[\left(S_{T_{i-1}}^{i-1} - K\right)^+\right] dK \tag{95}$$

$$= 2\int_0^\infty Black\left(K, S_{T_{i-1}}^{i-1}, V^M(K)\right) dK \tag{96}$$

where *Black* stands for the Black model.

For the consistent derivation of CMS convexity adjustment, volatility modelling is required. We use the SABR model, which is a market popular choice for swaption smile analysis, for the swap rate in order to infer from it the volatility smile surface to use for the integration above.

The SABR model assumes that $S_t^{i-1}$ evolves under the associated forward swap measure $Q_D^{t_{i-1}}$ according to:

$$dS_t^{i-1} = V_t (S_t^{i-1})^\beta dZ_t^{i-1} \tag{97}$$

$$dV_t = \epsilon V_t dW_t^{i-1} \tag{98}$$

$$V_0 = \alpha \tag{99}$$

Hagan et. al. (2002), derive the following approximation for the implied volatility of the swaption with maturity $T_{i-1}$:

$$\sigma^{imp}(K, S_t^{i-1}) \approx \frac{\alpha}{(S_t^{i-1}K)^{\frac{1-\beta}{2}}\left[1 + \frac{(1-\beta)^2}{24}\ln^2\left(\frac{S_t^{i-1}}{K}\right) + \frac{(1-\beta)^4}{1920}\ln^4\left(\frac{S_t^{i-1}}{K}\right)\right]}\frac{z}{x(z)} \tag{100}$$

$$\cdot\left\{1 + \left[\frac{(1-\beta)^2\alpha^2}{24(S_t^{i-1}K)^{1-\beta}} + \frac{\rho\beta\epsilon\alpha}{4(S_t^{i-1}K)^{\frac{1-\beta}{2}}} + \epsilon^2\frac{2-3\rho^2}{24}\right]T_{i-1}\right\} \tag{101}$$

where,

$$z := \frac{\epsilon}{\alpha}(S_t^{i-1}K)^{\frac{1-\beta}{2}}\ln\left(\frac{S_t^{i-1}}{K}\right) \tag{102}$$

and

$$x(z) := \ln\left\{\frac{\sqrt{1+2\rho z + z_2} + z - \rho}{1-\rho}\right\} \tag{103}$$

The above formula provides us with an efficient approximation for the SABR implied vilatility for each strike $K$. Now, we can calculate the CMS convexity adjustment by:

$$CA^{SABR}(S_t^{i-1};\delta) = \theta(S_t^{i-1})\left(\frac{2}{(S_t^{i-1})^2}\int_0^\infty Black\left(K, S_t^{i-1}, v^{imp}(K, S_t^{i-1})\right)dK - 1\right) \tag{104}$$

where,

$$v^{imp}(K, S_t^{i-1}) := \sigma^{imp}(K, S_t^{i-1})\sqrt{T_{i-1}} \tag{105}$$

We consider a different SABR model for each swap rate contained in the CMS payoff and we perform a calibration of all the SABR parameters (four parameters $(\alpha, \beta, \rho, \epsilon)$ for each swap rate) to swaption volatility smile and CMS spread quoted by the market. See Mercurio and Pallavicini (2006) for a detailed description of the calibration procedure.

# 5 Market Data

We used three data sets in this study, one containing Euro Money markets instruments for the construction of the yield curves, CMS swap spreads with maturity 5 years, where the associated underlying swaps have maturity 10 years (i.e. $X_{5,10}$), and swaption volatilities for different strikes, as well as implied Black ATM swaption volatilities. All market data were collected from Bloomberg.

Data sets are presented in detail below:

- For the discounting curve, we use EONIA fixing, OIS from 3 months to 30 years.
- For the 3m curve, we use EURIBOR three-months fixing, FRA rates up to 15m, and swaps from two to thirty years, paying an annual fix rate in exchange for the EURIBOR 3m rate.
- For the 6m curve, we use EURIBOR six-months fixing, FRA rates up to 2Y, and swaps from two to thirty years, paying an annual fix rate in exchange for the EURIBOR 6m rate.

The market quotes a value for the CMS spread which makes the CMS swap fair. However, it quotes the spread only for few CMS swap maturities and tenors (usually 5,10,15,20,30 years). In the Euro market, the CMS tenor is equal to three months, while the $c$-year IRS's used as indexation in the CMS has LIBOR payments of 6m or 1y frequency. Thus, CMS spreads depend on three different curves in our framework; the funding curve used to discount the cash flows of the CMS swap, which we consider to be the risk free curve (i.e. OIS curve); the 3m forwarding curve for the EURIBOR rates payed in the second leg of the CMS; and the 6m (or 1y) forwarding curve for the EURIBOR rates payed by the indexation IRS.

# 6 Numerical Calculations

We compare numerically the accuracy of the approximations for the CMS convexity adjustments, against the Black and SABR models convexity adjustments presented above. Our example is based on Euro data as of 03 February 2006.

We test a CMS with maturity 5 years (i.e. $n\delta = 5$), where the associated underlying swaps have maturity 10 years (i.e. $c\tau = 10$).

The closing price for the CMS spread with maturity 5 years, and associated underlying swaps with maturity 10 years, is $X_{5,10} = 64.9$ bps. The ATM swaption volatility is $\sigma_{5,10}^{ATM} = 0.15$, and swaption volatilities for different strikes are given below.

| Expiry | Tenor | -200 | -100 | -50 | -25 | 25 | 50 | 100 | 200 |
|---|---|---|---|---|---|---|---|---|---|
| 5y | 10y | 6.54% | 2.30% | 0.93% | 0.41% | -0.30% | -0.51% | -0.68% | -0.39% |

For the parameters of the term structure in case 2, we choose the values: $(a, b, k) = (0.01, 0.002, 0.1)$. Finally, when we apply the case with the spread, we assume that the spread is constant $X_t = 100$ bps, while its volatility is $\sigma_{t,x} = o.1$, and the correlation $\rho_{s,x} = 0.9$.

Our results are summarized in the following tables:

| | Market | Case 1 | Case 2 | SABR |
|---|---|---|---|---|
| Price | 0.00649 | 0.006237 | 0.006402 | 0.006406 |
| Difference (in bps) | | 2.53 | 0.89 | 0.83 |

**Table 1:** Prices of CMS spread for cases 1 and 2, and difference (in bps) between market and models - No spread

| | Market | Case 1 | Case 2 | SABR |
|---|---|---|---|---|
| Price | 0.00649 | 0.006352 | 0.0065001 | 0.006406 |
| Difference (in bps) | | 1.38 | 0.1 | 0.83 |

**Table 2:** Prices of CMS spread for cases 1 and 2, and difference (in bps) between market and models - when spread is incorporated

| i | CA (case 1) | CA (case 2) | CA (case 1 - with spread) | CA (case 2 - with spread) |
|---|---|---|---|---|
| 1 | 0.000036 | 0.000032 | 0.000037 | 0.000033 |
| 2 | 0.000074 | 0.000083 | 0.000076 | 0.000092 |
| 3 | 0.000114 | 0.000198 | 0.000116 | 0.000218 |
| 4 | 0.000154 | 0.000313 | 0.000158 | 0.000343 |
| 5 | 0.000196 | 0.000428 | 0.000201 | 0.000469 |
| 6 | 0.000238 | 0.000542 | 0.000245 | 0.000593 |
| 7 | 0.000282 | 0.000656 | 0.000289 | 0.000718 |
| 8 | 0.000326 | 0.000769 | 0.000335 | 0.000842 |
| 9 | 0.000373 | 0.000883 | 0.000383 | 0.000967 |
| 10 | 0.000420 | 0.000995 | 0.000431 | 0.001090 |
| 11 | 0.000468 | 0.001107 | 0.000480 | 0.001213 |
| 12 | 0.000514 | 0.001216 | 0.000528 | 0.001334 |
| 13 | 0.000564 | 0.001326 | 0.000579 | 0.001455 |
| 14 | 0.000615 | 0.001436 | 0.000631 | 0.001576 |
| 15 | 0.000667 | 0.001544 | 0.000685 | 0.001697 |
| 16 | 0.000721 | 0.001652 | 0.000739 | 0.001817 |
| 17 | 0.000775 | 0.001759 | 0.000795 | 0.001936 |
| 18 | 0.000831 | 0.001865 | 0.000852 | 0.002054 |
| 19 | 0.000888 | 0.001970 | 0.000911 | 0.002172 |
| 20 | 0.000946 | 0.002074 | 0.000970 | 0.002289 |

**Table 3:** Convexity adjustments for cases 1 and 2, with and without spread

# 7 Conclusions

We have developed a new CMS convexity adjustment by using different curves for discounting and forwarding. In the first case, the adjustment is derived under the assumption that the term structure of interest rates is flat and has only parallel shifts, while in the second case, the term structure may tilt. In both cases we have incorporated the spread between the OIS (discounting) curve and the swap (forwarding) curve, where we assume it to be flat and constant.

Compared to the first case, in the second case, the approximation has an additional term that depends upon the slope parameter $b$.

We have calibrated both convexity adjustments to real market data, by using swaption volatilities, and calculated the difference between market and model CMS spread. We compared our results with the smile-consistent CMS adjustment using the SABR model, which is

the market practice.

The second convexity adjustment is significantly larger than the first one.

In terms of the pricing of the CMS spread, we obtain a very good fit of market quotes in all cases. Especially when we use the convexity adjustment of case 2 (i.e. when the term structure tilts) and incorporate a constant spread, the quality of the fit is very good ($\sim 0.1$ bps) difference from the market price.

We aim to test our convexity adjustments to CMS spreads with longer maturities (i.e. 15 or 20 years), as well as for a bigger data sample, so that we can make a safe conclusion as to whether the CMS market, prices in a single or multi curve framework. We also aim to use historical volatility for the spread as well as the correlation.

Finally, the convexity adjustments we proposed can also be tested by trying to compute the true value of the convexity adjustment by simulation.

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