Bank Capital and Loan Takedown Risk

Bryan Stanhouse
and
Duane Stock

Division of Finance
Price College of Business
University of Oklahoma
Norman, Oklahoma

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During the financial crisis of 2008 not just one but many banks suffered sustained pressure on their reserves (primary and secondary) as credit lines were dramatically taken down by non-financial borrowers. As banks simultaneously sought to liquidate assets, intermediary losses were inevitable. For poorly capitalized intermediaries even a small loss on converted assets can lead to distress and potential insolvency. In an interconnected financial system, this can cause markets to freeze up and lead to severe repercussions for the rest of the economy. Because banks acting in their own best interests tend to pay too little attention to such "systemic risk," it becomes the responsibility of regulators to safeguard the functioning of the U.S. financial system. This paper investigates the relationship between intermediary capital and the stochastic usage of credit commitments under two circumstances. Our research provides insight into capital determination by profit maximizing intermediaries when they are, alternatively, constrained and unconstrained in their management of bank equity. Comparisons of the behavior of bank capital under these two schemes will hopefully lead to more effective regulation of our financial system by authorities.

Asset returns relative to the cost of acquiring permanent funding dictates the level of bank capital, while intertemporal bank revenues and funding costs clearly depend upon the nature of the stochastic takedown of the loan commitment. For example, if a credit line is entirely depleted within a few days of its inception then usage fees will generate sizable revenues while non-usage fees will be insignificant. The assets used to fund the loan drawdown will give rise to very little interest income and asset conversion costs will be fully realized. On the other hand, if a dramatic loan takedown occurs near the maturity of the credit line then the bank's usage fees will be small while the non-usage revenue will be relatively large. Assets, held to fund borrower

usage, will generate sizable interim income but suffer liquidation costs at the end of the loan contract. Or if the credit line goes nearly unused through the life of the loan commitment then usage fees will be negligible while the bank will enjoy increased income from non-usage fees. Interest earned from loan funding sources will be enhanced and asset liquidation costs will be small. While the relative attractiveness of the assets is shaped by the uncertain outcomes detailed above, the cost of acquiring permanent funding depends upon the deposit rate paid to account holders and the cost of bank capital.

We proxy the uncertain timing and usage of borrower credit lines over time with trended Brownian motion. Relying on "time to first passage" mathematics, we derive a probability density function for the time to depletion of the bank credit line as well as the likelihoods for the time to exhausting the sources of liquidity that fund the loan takedown. Armed with these analytical results, we solve for the optimal level of bank capital within a simultaneous equation framework in order to capture the interrelationship of the endogenous variables along with the impact of the exogenous variates. We believe that while capital determines the scale of bank operations there exists a delicate balance between equity, reserves, treasury holdings and bank credit lines. If we ignore these interdependencies we forfeit the opportunity to understand the complementary or substitutionary effects of bank decision variables. Consequently, we solve for the level of capital as well as reserves, treasuries and the size of loan commitments that maximize the intermediary's intertemporal profits. The optimality conditions produce a system of integral differential equations which refuse to yield reduced form solutions and provide no immediate intuition. Therefore, the maximizing values of the bank's decision variables were simulated over a host of realistic scenarios. Our research produces four tables of simulated

optimal values for bank capital and of the intermediary's assets. Table 1 documents the comparative static behavior of the intermediary's decision variables when equity is unencumbered by capital requirements while the second table examines the impact of the same parametric changes on bank behavior when equity is a fixed proportion of lending. The remaining tables document the expected time to liquidity depletion under different capital requirement schemes.[1]

The dramatic loan takedowns of 2008 demand that the determination of optimal bank capital acknowledge that loan takedowns are uncertain. By understanding the recalibrations of profit maximizing equity, in this context, authorities will be in a better position to influence and regulate banks during a financial crisis. For instance, the size of the Federal Reserve's balance sheet has tripled since its pre-recession level to nearly $2.9T. As part of their "Quantitative Easing" programs, the Fed has eagerly bought up treasury securities and pushed debt returns to historic lows throughout the term structure. Our simulations show that QE1 and QE2 have surprising consequences for bank capital and an impact upon intermediary lending. The average weekly bank deposit growth rate for the five years prior to the Lehman brothers failure, in September of 2008, was .095%. For the 14 days following the Lehman closing the appreciation in deposits was 10 times larger. Much like this surge in permanent funding, our results record the impact of deposit re-intermediation upon bank capital and the configuration of intermediary assets.[2] The "Troubled Asset Relief Program" (TARP) was passed by an act of Congress in October of 2008 with a budget that amounted to $700B. To the dismay of many economists, no strings were attached to the $300B that were used for bank capital infusions. In fact, very little of the $0.3T was used to support commercial bank lending. Our simulations detail just what changes in bank assets regulators can reasonably expect from equity

enhancements in an environment of uncertain credit line usage. Since the summer of 2010 the topic of the Federal Reserve paying interest on bank reserves (IOR) has been hotly debated. Many economists have argued that if the Fed were to eliminate "IOR" banks would be forced to lend more since intermediaries would have one less source of income. Others maintain that paying interest on bank reserves actually encourages lending. Our endogenous capital approach to this issue provides an unambiguous and insightful resolution to the "IOR" controversy.

2. Time to loan depletion

For our model, we will assume that the cash needs of the borrower $C(t)$ follow a Brownian motion process with a drift of μ where $\mu > 0$. [3] The borrower's cash needs represent a dollar for dollar takedown of the credit line and the time to loan depletion T_W of $C(t)$ is characterized by $C(T_W) = W$ where W is the size of the loan commitment.[4] If $C(t) < W$ then $t < T_W$. That is, we let $p(c_0, c; t)$ be the probability that $C(t) = c$ and that the process does not reach the point of loan depletion in the time interval (0,t). Consequently,

$$P\{C(t) < W \text{ for } t < T_W \mid C(0) = c_0\} = \int_{-\infty}^{W} p(c_0, c; t)\, dc \equiv P(c_0, W; t) \tag{1a}$$

is the probability that the time to depletion of W has not occurred by t, we have

$$P(c_0, W; t) = prob(T_W \geq t). \tag{1b}$$

So that

$$P(c_0, W; t) = \int_{-\infty}^{W} p(c_0, c; t)\, dc = 1 - G(t; c_0, W) \tag{1c}$$

where $G(t; c_0, W)$ is the cumulative probability of the time to the loan's depletion.

Consequently,

$$\frac{\partial P(c_0,W;t)}{\partial t} = -G'(t;c_0,W) \tag{2a}$$

or

$$-\frac{\partial P(c_0,W;t)}{\partial t} = g(t;c_0,W) \tag{2b}$$

where $g(t;c_0,W)$ is the marginal probability density function of T_W.

To solve for $g(t;c_0,W)$ we simply take advantage of the fact that since $p(c_0,c;t)$ is Brownian motion it satisfies Kolmogorov's diffusion equation.[5] Therefore $P(c_0,W;t)$

satisfies Kolmogorov's equation and, consequently, $g(t;c_0,W)$ must satisfy the same characterization, so we have

$$\frac{\partial g(t;c_0,W)}{\partial t} = \mu \frac{\partial g(t;c_0,W)}{\partial c_0} + \frac{1}{2}\sigma^2 \frac{\partial^2 g(t;c_0,W)}{\partial c_0 \partial c_0} \tag{3}$$

where σ is the instantaneous standard deviation of the borrower's cash needs.[6] The expression above is a partial differential equation whose direct solution is a bit messy.[7] It is far more convenient to solve for $g(t;c_0,W)$ by using a Laplace transform with respect to time.[8]

Taking a Laplace transformation of (3) will convert the original partial differential equation involving the borrower's cash needs, coordinate "c", and the time coordinate "t" into an ordinary differential equation in "c". Thus to begin with we have

$$\int_0^\infty e^{-st} \frac{\partial g(t;c_o,W)}{\partial t} dt = \int_0^\infty e^{-st} \left[\mu \frac{\partial g(t;c_o,W)}{\partial c_o} + \frac{1}{2}\sigma^2 \frac{g(t;c_o,W)}{\partial c_o \partial c_o} \right] dt. \tag{4}$$

A Laplace transform of a linear combination of functions is the linear combination of the transforms and since both μ and σ^2 pass through the integral operator we have

$$\frac{\partial g^*}{\partial t} = \mu \frac{\partial g^*}{\partial c_0} + \frac{1}{2}\sigma^2 \frac{\partial^2 g^*}{\partial c_0 \partial c_0}$$

$$\text{where } g^* = \int_0^\infty e^{-st} g(t;c_0,W) dt.$$

Furthermore,

$$sg^* = \mu \frac{\partial g^*}{\partial c_0} + \frac{1}{2}\sigma^2 \frac{\partial^2 g^*}{\partial c_0 \partial c_0}$$

because differentiation of the transform g^* with respect to "t" corresponds simply to the multiplication of g^* by s.

We have a second order linear differential equation with constant coefficients

$$s\lambda = \mu \frac{\partial \lambda}{\partial c_0} + \frac{1}{2}\sigma^2 \frac{\partial^2 \lambda}{\partial c_0 \partial c_0} \tag{5a}$$

where $\lambda = g^*$.

Adopting the trial solution of (5a) as $\lambda(c_0) = Ae^{\theta c_0}$ yields $\lambda'(c_0) = \theta Ae^{\theta c_0}$ and $\lambda''(c_0) = \theta^2 Ae^{\theta c_0}$ so that (5a) becomes

$$sAe^{\theta c_0} = \mu\theta Ae^{\theta c_0} + \frac{1}{2}\sigma^2\theta^2 Ae^{\theta c_0}$$

or

$$Ae^{\theta c_0}(\mu\theta + \frac{1}{2}\sigma^2\theta^2 - s) = 0$$

or

$$Ae^{\theta c_0}(\frac{1}{2}\sigma^2\theta^2 + \mu\theta - s) = 0.$$

Solving $(\frac{1}{2}\sigma^2\theta^2 + \mu\theta - s) = 0$ for θ using the quadratic formula yields

$$\theta_1(s) \text{ and } \theta_2(s) = \frac{-\mu \mp \sqrt{\mu^2 + 2s\sigma^2}}{\sigma^2} \quad . \qquad (5b)$$

The general solution to (5a) is

$$\lambda(c_0) = A_1 e^{\theta_1 c_0} + A_2 e^{\theta_2 c_0}. \qquad (6)$$

Note that if we take the positive square root in (5b), for real values of s we have

$$\theta_1(s) < 0 < \theta_2(s) \quad , \qquad (s > 0) .$$

In (6), it is clear that A_1 must be equal to zero because otherwise $A_1 e^{\theta_1 c_0}$ would be unbounded as c_0 approaches $-\infty$ with $\theta_1(s)$ less than 0. Furthermore $A_2 e^{\theta_2 c_0}$ should be written as $e^{\theta_2 (c_0 - W)}$ so that when $c_0 = W$, $\lambda(W)$ equals one.[9] With $\lambda(W) = 1$, the inverse function $g(s; c_0, W)$ will be zero.[10] Consequently, the likelihood of $T_W > 0$ is zero when $c_0 = W$ and establishes the fact that loan depletion will take place immediately.

Finally, we have

$$\lambda(c_0) = g^*(s; c_0, W) = e^{(c_0 - W)\theta_2(s)}$$

or

$$\lambda(c_0) = g^*(s; c_0, W) = e^{-(c_0 - W)(\mu - (\mu^2 + 2s\sigma^2)^{1/2})/\sigma^2} \tag{7}$$

for $c_0 < W$. We conjecture that

$$\mathrm{L}[\mathrm{g}(\mathrm{t}; \mathrm{c}_0, W); s] = e^{-(c_0 - W)\{\mu - (\mu^2 + 2s\sigma^2)^{1/2}\}/\sigma^2} \tag{8a}$$

where L represents the Laplace operator and

$$g(t;c_0,W)=\frac{(W-c_0)}{\sigma\sqrt{2\pi t^3}}\exp[\frac{-(W-c_0-\mu t)^2}{2\sigma^2 t}].$$

$\mathrm{L}[\mathrm{g}(\mathrm{t};\mathrm{c}_0,W);s]$ could be written as

$$\mathrm{L}[\mathrm{g}(\mathrm{t};\mathrm{c}_0,W);s]=\frac{(W-c_0)}{\sigma\sqrt{2\pi}}\int_0^\infty\frac{1}{\sqrt{t^3}}e^{\frac{-[(W-c_0)^2+\mu^2t^2-2(W-c_0)\mu t]}{2\sigma^2 t}}e^{-st}dt \tag{8b}$$

or as

$$\mathrm{L}[\mathrm{g}(\mathrm{t};\mathrm{c}_0,W);s]=\frac{(W-c_0)}{\sigma\sqrt{2\pi}}e^{\frac{2(W-c_0)\mu}{2\sigma^2}}\int_0^\infty\frac{1}{\sqrt{t^3}}e^{\frac{-[(W-c_0)^2+\mu^2t^2+2s\sigma^2t^2]}{2\sigma^2 t}}dt \tag{8c}$$

or as

$$\mathrm{L}[\mathrm{g}(\mathrm{t};\mathrm{c}_0,W);s]=\frac{(W-c_0)}{\sigma\sqrt{2\pi}}e^{\frac{(W-c_0)\mu}{\sigma^2}}\int_0^\infty\frac{1}{\sqrt{t^3}}e^{\frac{-(W-c_0)^2}{2\sigma^2}t^{-1}-[\frac{\mu^2+2s\sigma^2}{2\sigma^2}]t}dt \tag{8d}$$

or as

$$\mathrm{L}[\mathrm{g}(\mathrm{t};\mathrm{c}_0,W);s]=e^{\frac{(W-c_0)\mu}{\sigma^2}}\frac{(W-c_0)}{\sigma\sqrt{2\pi}}\int_0^\infty\frac{1}{\sqrt{t^3}}e^{-qt^{-1}-\rho t}dt \tag{8e}$$

where $q=\frac{(W-c_0)^2}{2\sigma^2}$, $\rho=(\frac{\mu^2+2s\sigma^2}{2\sigma^2})$.

Using the mathematical fact that

$$\int_0^\infty \mathrm{t}^{-3/2} e^{-q/t} e^{-\rho t} dt = \sqrt{\frac{\pi}{q}} e^{-2\sqrt{q\rho}}$$

allows us to rewrite (8e) as

$$\mathrm{L}[\mathrm{g}(\mathrm{t};\mathrm{c}_0, W); s] = e^{\frac{(W-c_0)\mu}{\sigma^2}} \cdot \frac{(W-c_0)}{\sigma\sqrt{2\pi}} \cdot \frac{\sqrt{\pi}}{\frac{(W-c_0)}{\sqrt{2\sigma^2}}} e^{-2\frac{(W-c_0)}{\sqrt{2\sigma^2}} \cdot \frac{\sqrt{\mu^2+2s\sigma^2}}{\sqrt{2\sigma^2}}} \tag{8f}$$

$$\mathrm{L}[\mathrm{g}(\mathrm{t};\mathrm{c}_0, W); s] = e^{\frac{\mu(W-c_0)}{\sigma^2}} e^{-[\frac{(W-c_0)(\mu^2+2s\sigma^2)^{1/2}}{\sigma^2}]} \tag{8g}$$

$$\mathrm{L}[\mathrm{g}(\mathrm{t};\mathrm{c}_0, W); s] = e^{\frac{1}{\sigma^2}(W-c_0)[\mu-(\mu^2+2s\sigma^2)^{1/2}]} \quad . \tag{8h}$$

So indeed our conjecture is correct and the Laplace transform of

$\mathrm{g}(\mathrm{t};\mathrm{c}_0, W) = \frac{(W-c_0)}{\sigma\sqrt{2\pi t^3}} \exp[\frac{-(W-c_0-\mu t)^2}{2\sigma^2 t}]$ is the right hand side of (8a).

The expression $\mathrm{g}(\mathrm{t};\mathrm{c}_0, W)$ is then the probability density function of the time to loan depletion.[11] This probability density function integrates to one and the time to loan depletion has a mean of $(W-c_0)/\mu$.[12]

3. Intertemporal revenues and costs

Optimal capital equates the expected marginal revenue of bank assets to the expected marginal costs of the intermediary's permanent funding. The left hand side of the bank's balance sheet consists of reserves (R), government securities (G) and loans (L). While all three of the assets provide revenue to the intermediary, it is also true that R and G are sources of liquidity for the bank as it finances the uncertain takedown of borrower credit lines. If loan draw down exhausts both of these assets our model acknowledges the possibility of interbank borrowing (B).

A characterization of expected intertemporal revenues is nearly unmanageable but it provides the most important part of the framework in which to consider the bank's optimization problem. In regard to its intertemporal credit line revenues, the bank must consider two eventualities in four alternative time periods. The simplest scenario is where the loan recipient draws down the entire line in the first quarter of his year-long loan commitment.

The bank's earnings associated with a first quarter loan depletion would be

$$(\phi+\Gamma_1)\cdot\left\{\int_0^{\bar{t}_1}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)dc\right]ds\right\}+W(\phi+\Gamma_1)\left[\int_0^{1/4}dt-\int_0^{1/4}t\frac{g(t)}{\int_0^{1/4}g(t)dt}dt\right]+W(\phi+\Gamma_2)\left[\int_{1/4}^{1/2}dt\right]$$

$$+\ W(\phi+\Gamma_3)\left[\int_{1/2}^{3/4}dt\right]+W(\phi+\Gamma_4)\left[\int_{3/4}^{1}dt\right] \qquad (9)$$

where $\bar{t}_1 = \int_0^{1/4} t \frac{g(t)}{\int_0^{1/4} g(t)dt} dt$ and $g(t) = \frac{W}{\sigma\sqrt{2\pi t^3}} e^{-W-\mu t^2/2\sigma^2 t}$.

The innermost integral in the first term of the expression determines the mean cash needs of the borrower $\int_{-\infty}^{\infty} cp(c_0, c; s)dc$ at a point in time *s*. The next integral $\int_0^{\bar{t}_1} \cdot ds$ accumulates average borrower needs from the inception of the loan until the time the line is completely drawn down. We characterize the upper limit as being $\bar{t}_1$ which is the expected time to exhaustion of the credit line presuming that the loan is completely depleted within the first 3 months of being extended. The conditional mean time to W, $\bar{t}_1$ is mathematically given by $\int_0^{1/4} t \frac{g(t)}{\int_0^{1/4} g(t)dt} dt$ according to Bayes.

Of course, $\phi + \Gamma_1 \cdot \left\{ \int_0^{\bar{t}_1} \left[\int_{-\infty}^{\infty} cp(c_0, c; s)dc \right] ds \right\}$ is the amount of the bank's earnings in the first quarter up to the time the credit line is exhausted. The bank's earnings on the loan in the first quarter after the line is completely taken down is given by

$$W(\phi + \Gamma_1) \left[\int_0^{1/4} dt - \int_0^{1/4} t \frac{g(t)}{\int_0^{1/4} g(t)dt} dt \right].$$

The three remaining terms $W(\phi+\Gamma_2)\int_{1/4}^{1/2} dt + W(\phi+\Gamma_3)\int_{1/2}^{3/4} dt + W(\phi+\Gamma_4)\int_{3/4}^{1} dt$ in (9) document the bank's earnings in the final nine months of the contract's existence. The expected earnings attributable to a loan that is exhausted in its first quarter must be adjusted by the likelihood that W is reached by borrower needs within the first three months, $\int_0^{1/4} g(t)dt$, after his final statistical acknowledgement, we have

$$(\phi+\Gamma_1)\cdot\left\{\int_0^{\bar{l}_1}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)dc\right]ds\right\}\cdot\left\{\int_0^{1/4} g(t)dt\right\} + W(\phi+\Gamma_1)\left[\int_0^{1/4} dt - \int_0^{1/4} t\frac{g(t)}{\int_0^{1/4} g(t)dt}dt\right]\cdot\left\{\int_0^{1/4} g(t)dt\right\}$$

$$+W(\phi+\Gamma_2)\left[\int_{1/4}^{1/2} dt\right]\cdot\left\{\int_0^{1/4} g(t)dt\right\} + W(\phi+\Gamma_3)\left[\int_{1/2}^{3/4} dt\right]\cdot\left\{\int_0^{1/4} g(t)dt\right\}$$

$$+W(\phi+\Gamma_4)\left[\int_{3/4}^{1} dt\right]\cdot\left\{\int_0^{1/4} g(t)dt\right\}. \qquad (10)$$

The contribution to intertemporal expected earnings of credit line loans that are, respectively, depleted in the second quarter, the third quarter and the fourth quarter follow the analysis just provided and are listed below.

Expected Revenues Associated with a 2nd Quarter Depletion of the Loan

$$(\phi+\Gamma_1)\cdot\left\{\int_0^{1/4}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc]\;ds\right]\right\}\cdot\left\{\int_{1/4}^{1/2} g(t)dt\right\}$$

$$+(\phi+\Gamma_2)\cdot\left\{\int_{1/4}^{\bar{t}_2}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]ds\right\}\cdot\left\{\int_{1/4}^{1/2} g(t)dt\right\}+W(\phi+\Gamma_2)\left[\int_0^{1/2} dt-\int_{1/4}^{1/2} t\,\frac{g(t)}{\int_{1/4}^{1/2} g(t)dt}\,dt\right]\cdot\left\{\int_{1/4}^{1/2} g(t)dt\right\}$$

$$+W(\phi+\Gamma_3)\left[\int_{1/2}^{3/4} dt\right]\cdot\left\{\int_{1/4}^{1/2} g(t)dt\right\}+\;W(\phi+\Gamma_4)\left[\int_{3/4}^{1} dt\right]\cdot\left\{\int_{1/4}^{1/2} g(t)dt\right\}$$

where $\bar{t}_2 = \int_{1/4}^{1/2} t\,\frac{g(t)}{\int_{1/4}^{1/2} g(t)\,dt}\,dt$. (11)

Expected Revenues Associated with a 3rd Quarter Depletion of the Loan

$$(\phi+\Gamma_1)\cdot\left\{\int_0^{1/4}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]ds\right\}\cdot\left\{\int_{1/2}^{3/4} g(t)dt\right\}+(\phi+\Gamma_2)\cdot\left\{\int_{1/4}^{1/2}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]dc\right\}\cdot\left\{\int_{1/2}^{3/4} g(t)dt\right\}$$

$$+(\phi+\Gamma_3)\left\{\int_{1/2}^{\bar{t}_3}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]ds\right\}\left\{\int_{1/2}^{3/4} g(t)dt\right\}+W(\phi+\Gamma_3)\left[\int_0^{3/4} dt-\int_{1/2}^{3/4} t\frac{g(t)}{\int_{1/2}^{3/4} g(t)dt}dt\right]\left\{\int_{1/2}^{3/4} g(t)dt\right\}$$

$$+W(\phi+\Gamma_4)\cdot\left[\int_{3/4}^{1} dt\right]\cdot\left\{\int_{1/2}^{3/4} g(t)dt\right\} \quad \text{where} \quad \bar{t}_3=\int_{1/2}^{3/4} t\frac{g(t)}{\int_{1/2}^{3/4} g(t)\,dt}dt\ . \tag{12}$$

Expected Revenues Associated with a 4th Quarter Depletion of the Loan

$$(\phi+\Gamma_1)\cdot\left\{\int_0^{1/4}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]ds\right\}\cdot\left\{\int_{3/4}^{1} g(t)dt\right\}+(\phi+\Gamma_2)\cdot\left\{\int_{1/4}^{1/2}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]ds\right\}\cdot\left\{\int_{3/4}^{1} g(t)dt\right\}$$

$$+(\phi+\Gamma_3)\cdot\left\{\int_{1/2}^{3/4}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]ds\right\}\cdot\left\{\int_{3/4}^{1} g(t)dt\right\}+(\phi+\Gamma_4)\cdot\left\{\int_{3/4}^{\bar{t}_4}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]ds\right\}\cdot\left\{\int_{3/4}^{1} g(t)dt\right\}$$

$$+\ W(\phi+\Gamma_4)\cdot\left[\int_0^{1} dt-\int_{3/4}^{1} t\frac{g(t)}{\int_{3/4}^{1} g(t)dt}dt\right]\left\{\int_{3/4}^{1} g(t)dt\right\}\ \text{where}\ \bar{t}_4=\int_{3/4}^{1} t\frac{g(t)}{\int_{3/4}^{1} g(t)\,dt}dt\ . \tag{13}$$

Expected Revenues if the Loan Takedown is Partial

Of course, it is always possible that the borrower does not deplete the loan account before the loan commitment contract matures. In this case, the revenue in each quarter would be the product of the takedown rate $(\phi+\Gamma_i)$ and the mean takedown in each quarter, and for four quarters the bank would have

$$(\phi+\Gamma_1)\cdot\int_0^{1/4}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)dc\right]ds+(\phi+\Gamma_2)\cdot\int_{1/4}^{1/2}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)dc\right]ds$$

$$+(\phi+\Gamma_3)\cdot\int_{1/2}^{3/4}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)dc\right]ds+(\phi+\Gamma_4)\cdot\int_{3/4}^{1}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)dc\right]ds.$$

Adjusting the revenues by the likelihood that loan recipient's cash needs will not reach W means that the contribution of this scenario to the bank's intertemporal expected profits would be

$$\left\{\begin{array}{l}(\phi+\Gamma_1)\cdot\int_0^{1/4}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)dc\right]ds+(\phi+\Gamma_2)\cdot\int_{1/4}^{1/2}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)dc\right]ds \\ \qquad +(\phi+\Gamma_3)\cdot\int_{1/2}^{3/4}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)dc\right]ds+(\phi+\Gamma_4)\cdot\int_{3/4}^{1}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)dc\right]ds\end{array}\right\}\left\{1-\int_0^{T_M} g(t)dt\right\} \quad (14)$$

where $\left\{1-\int_0^{T_M} g(t)dt\right\}$ is the probability that the loan will not be completely drawn down by T_M.

Expected Revenues Associated with Loan Non-Usage Fees

The bank's expected earnings from non-usage fees should be considered under two circumstances.[13] The first situation is when the credit line is depleted before the bank's commitment to the borrower expires which has a likelihood of $\int_0^{T_M} g(t)dt$. The second scenario is characterized by the loan not being used up before it matures at T_M, which is probabilistically given as $1-\int_0^{T_M} g(t)dt$. If the loan is not completely taken down then the bank's expected lending would be given by the borrower's average cash needs from 0 to T_M or

$$\int_0^{T_M}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]ds \ .$$

The unused amount of the loan would be $\int_0^{T_M}\left\{W-\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]\right\}ds$. The expected revenue generated by a non-usage fee of r_N when the line is not exhausted, adjusted by the likelihood that the borrower's cash needs do not reach W , would be given as

$$r_N \cdot \int_0^{T_M}\left\{W-\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]\right\}ds \cdot \left[1-\int_0^{T_M} g(t)dt\right] .$$

If the loan is completely taken down before T_M, then usage up to complete depletion would be characterized as

$$\left\{ \int_0^{\bar{t}_W} \left[\int_{-\infty}^{\infty} cp(c_o, c; s)dc \right] ds \right\}$$

where

$$\bar{t}_w = \int_0^{T_M} t \frac{g(t)}{\int_0^{T_M} g(t)dt} dt .$$

The amount of the loan that would be considered unused until the time of depletion would be

$$\int_0^{\bar{t}_W} \left\{ W - \left[\int_{-\infty}^{\infty} cp(c_o, c; s)dc \right] \right\} ds$$

so that the expected earnings from the non-usage fee when the line is taken down before T_M and adjusted by the likelihood of a complete takedown would be

$$r_N \cdot \int_0^{\bar{t}_W} \left\{ W - \left[\int_{-\infty}^{\infty} cp(c_0, c; s)\, dc \right] \right\} ds \cdot \int_0^{T_M} g(t)dt \ .$$

Combining the expected value of the two scenarios that provide fees from non-usage would yield:

$$r_N \cdot \int_0^{T_M} \left\{ W - \left[\int_{-\infty}^{\infty} cp(c_0, c; s)\, dc \right] \right\} ds \cdot \left[1 - \int_0^{T_M} g(t) dt \right]$$

$$+ r_N \cdot \int_0^{\tilde{t}_W} \left\{ W - \left[\int_{-\infty}^{\infty} cp(c_0, c; s)\, dc \right] \right\} ds \cdot \int_0^{T_M} g(t) dt \quad . \qquad (15)$$

Expected revenues associated with loan funding sources

In conjunction with the determination of the optimal size of a loan commitment, the internal liquidity for funding the takedown must be configured to maximize the bank's intertemporal profits. The assets are denoted as R and G, they each enjoy rates of return (r_1, r_2) while only G has variable and fixed costs of conversion, vc_G and FC_G.[14] Once again, R and G are bank reserves and treasuries which are individually constrained to be non-negative and when taken with loans, must add up to the RHS of the bank's tee account.

Articulating the revenues associated with holding R to help fund the borrower's drawdown of his credit line involves the consideration of two circumstances. The first case is when R is entirely depleted by the takedown before T_M which has likelihood of

$$\int_0^{T_M} g_R(t) dt$$

where

$$g_R(t) = \frac{R - 0}{\sigma \sqrt{2\pi t^3}} e^{-(R-0-\mu t)^2 / 2\sigma^2 t} \, .$$ [15]

In this situation, the depletion of the holdings of R over time would correspond to the expected cash needs of the borrower from *0* to $\overline{t}_R$ given as

$$\int_0^{\overline{t}_R}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)dc\right]ds$$

where $$\overline{t}_R = \int_0^{T_M} t\,\frac{g_R(t)}{\int_0^{T_M} g_R(t)dt}\,dt.$$

The dollar for dollar depletion of R to fund the usage of the credit line means that the expected earnings of reserves when this account is exhausted before T_M would be given as

$$r_1 \cdot \int_0^{\overline{t}_R}\left\{R-\left[\int_{-\infty}^{\infty} cp(c_0,c;s)dc\right]\right\}ds \cdot \int_0^{T_M} g_R(t)dt.$$

If R is not completely depleted by the borrower's takedown before T_M then the expected earnings associated with R would be

$$r_1 \cdot \int_0^{T_M}\left\{R-\left[\int_{-\infty}^{\infty} cp(c_0,c;s)dc\right]\right\}ds \cdot \left[1-\int_0^{T_M} g_R(t)dt\right].$$

Combining the conditional expected values yields expected earnings for R as

$$r_1 \cdot \int_0^{\bar{t}_R} \left\{ R - \left[\int_{-\infty}^{\infty} cp(c_0, c; s)dc \right] \right\} ds \cdot \left[\int_0^{T_M} g_R(t)dt \right]$$

$$+ r_1 \cdot \int_0^{T_M} \left\{ R - \left[\int_{-\infty}^{\infty} cp(c_0, c; s)dc \right] \right\} ds \cdot \left[1 - \int_0^{T_M} g_R(t)dt \right]. \qquad (16)$$

Detailing the revenues associated with using G, a second source of immediate credit line funding, involves just one source of uncertainty but two thresholds. If R is not entirely depleted over the life of the loan commitment then government securities would earn $r_2 G \cdot \int_0^{T_M} dt$ and the contribution of this eventuality to the expected earnings of G would be

$$r_2 G \cdot \int_0^{T_M} dt \cdot \left[1 - \int_0^{T_M} g_R(t)dt \right] \qquad (17)$$

since $\left[1 - \int_0^{T_M} g_R(t)dt \right]$ is the likelihood of this scenario.

If R is exhausted funding the loan but G is not used up by the takedown the contribution to expected earnings of government securities would be

$$\left\{ r_2 G \cdot \int_0^{\bar{t}_{x1}} dt + r_1 \cdot \int_{\bar{t}_R}^{T_M} \left\{ R + G - \left[\int_{-\infty}^{\infty} cp(c_0, c; s) dc \right] \right\} ds \right\} \cdot \left[\int_0^{T_M} g_R(t) dt - \int_0^{T_M} g_{R+G}(t) dt \right]. \qquad (18)$$

where

$$g_{R+G}(t) = \frac{R + G - 0}{\sigma\sqrt{2\pi t^3}} e^{-(G+R-0-\mu t)^2 / 2\sigma^2 t}.$$

The first term in (18) accounts for the earnings on G until the time R is entirely depleted by the loan recipient's takedown at $\bar{t}_R$. The second term in the expression above accounts for the expected earnings attributable to G after it has been converted to reserves but before the cash needs of the borrower has depleted the account. The expression $\left[\int_0^{T_M} g_R(t) dt - \int_0^{T_M} g_{R+G}(t) dt \right]$ is the probability that the takedown will exceed R but fall short of (R+G).

Earnings on G could arise from the situation where the stochastic takedown of the loan exhausts both R and G before T_M. This possibility has a likelihood of $\int_0^{T_M} g_{R+G}(t)dt$. The expected value of this scenario to the bank's intertemporal profits would be

$$r_2 G \cdot \int_0^{\bar{t}_R} dt \cdot \int_0^{T_M} g_{R+G}(t)dt$$

$$+ r_1 \cdot \int_{\bar{t}_R}^{\bar{t}_{R+G}} \left\{ R + G - \left[\int_{-\infty}^{\infty} cp(c_0, c; s)dc \right] \right\} ds \cdot \int_0^{T_M} g_{R+G}(t)dt \qquad (19)$$

$$\text{where} \quad \bar{t}_{R+G} = \int_0^{T_M} t \frac{g_{R+G}(t)}{\int_0^{T_M} g_{R+G}(t)dt} dt.$$

Expected costs associated with funding the loan takedown

Articulating the costs associated with financing a loan takedown using government securities involves the consideration of three scenarios. If R is not entirely depleted before T_M then G would incur no conversion costs since it would not be used for funding. A second scenario occurs when R is completely depleted providing takedown funding while G is not exhausted. In this case, the bank would have expected conditional costs of

$$\left\{ \nu c_G \left[\int_{-\infty}^{\infty} cp(c_0, c; T_M)dc - \int_{-\infty}^{\infty} cp(c_0, c; \bar{t}_R)dc \right] + FC_2 \right\} \cdot \left[\int_0^{T_M} g_R(t)dt - \int_0^{T_M} g_{R+G}(t)dt \right]. \qquad (20)$$

The first term within the brackets, νc_G, details the per dollar costs of converting government securities into reserves for funding the takedown. The second term provides the expected amount of government securities that will need to be liquidated by determining the borrower's expected cash needs between $\bar{t}_R$ and T_M. The term FC_G proxies the fixed costs associated with converting G into reserves. The expression $\left[\int_0^{T_M} g_R(t)dt - \int_0^{T_M} g_{R+G}(t)dt\right]$ is the probability that the takedown will exceed R but fall short of (R+G) before T_M.

The final scenario takes place when both R and G are completely exhausted by the credit line being taken down. In this case, conditional expected costs would be characterized by

$$\left\{\nu c_G \cdot \left[\int_{-\infty}^{\infty} cp(c_0, c; \bar{t}_{R+G})dc - \int_{-\infty}^{\infty} cp(c_0, c; \bar{t}_R)dc\right] + FC_G\right\} \cdot \int_0^{T_M} g_{R+G}(t)dt.^{16} \qquad (21)$$

The likelihood that borrower takedown will exceed (R+G) before the loan commitment matures is given by $\int_0^{T_M} g_{R+G}(t)dt.$[17]

The expected intertemporal funding costs must account for the possibility that the loan takedown could exceed the value of the assets immediately available for funding (R+G). If the credit line is completely drawn down before T_M then the impact of borrowing by the bank could be described by the following conditional expected cost

$$\left\{ r_3 \cdot \left[\int_{\bar{t}_{R+G}}^{\bar{t}_W} \left[\int_{-\infty}^{\infty} cp(c_0, c; s)dc \right] - (R+G) \right] ds + FC_B \right\} \cdot \left\{ \int_0^{T_M} g(t)dt \right\}$$ [18]

$$+ \left\{ r_3 \cdot W - (R+G) \cdot \left\{ T_M - \int_{\bar{t}_{R+G}}^{T_M} t \frac{g(t)}{G(T_M)} dt \right\} \cdot \int_0^{T_M} g(t)dt \right\}. \qquad (22)$$

The first term above records the interbank borrowing rate, r_3, that the bank must pay for emergency money to fund the loan takedown.[19] The expression

$$\int_{\bar{t}_{R+G}}^{\bar{t}_W} \left\{ \left[\int_{-\infty}^{\infty} cp(c_0, c; s)dc \right] - (R+G) \right\} ds$$

accounts for the amount of funds needed to bridge the gap between expected borrower needs and (R+G) from $\bar{t}_{R+G}$ to $\bar{t}_W$.[20]

Alternatively, if the credit line is not depleted before T_M but the expected takedown does exceed (R+G) then the conditional expected cost would be given by

$$\left\{ r_3 \cdot \int_{\bar{t}_{R+G}}^{T_M} \left\{ \left[\int_{-\infty}^{\infty} cp(c_0, c; s)dc \right] - (R+G) \right\} ds + FC_B \right\} \cdot \left\{ 1 - \int_0^{T_M} g(t)dt - \int_0^{T_M} g_{R+G}(t)dt \right\}. \qquad (23)$$

Costs associated with permanent asset funding

Our model recognizes that the bank's supply of equity and cost of equity logically depends upon the amount of capital raised where the more capital that is raised, the greater the unit cost. This is consistent with Kopecky and VanHoose (2004) who suggest the cost of equity is a positive quadratic function of the amount of bank equity raised. Furthermore, the idea the cost of equity rises with the quantity raised is consistent with Asquith and Mullins (1986) who find that seasoned equity offerings reduce the share prices of the issuing firm. They suggest the reduction in share price is consistent with a negative sloping demand schedule where the firm supplying more shares has to reduce the price of a share to induce an increase in the quantity of equity provided. The idea of a negative sloping demand for equity schedule is well accepted. For example, see Gao and Ritter (2010) who review many articles supporting the idea of negatively sloped demand schedule. In fact, Gao and Ritter (2010) find that alternative techniques for marketing seasoned equity clearly affects the severity of the negative slope of the demand for equity schedule.

4. The Objective Function

Combining expressions (10), (11), (12), (13), (14), (15), (16), (17), (18), (19), (20), (21), (22) and (23) yields the bank's expected intertemporal profit from a credit line, $E(\pi)$:

$$(\phi+\Gamma_1)\cdot\left\{\int_0^{\tilde{t}_1}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]ds\right\}\cdot\left\{\int_0^{1/4} g(t)dt\right\}+W(\phi+\Gamma_1)\left[\int_0^{1/4} dt-\int_0^{1/4} t\,\frac{g(t)}{\int_0^{1/4} g(t)dt}\,dt\right]\cdot\left\{\int_0^{1/4} g(t)dt\right\}$$

$$+W(\phi+\Gamma_2)\left[\int_{1/4}^{1/2} dt\right]\cdot\left\{\int_0^{1/4} g(t)dt\right\}+\ W(\phi+\Gamma_3)\left[\int_{1/2}^{3/4} dt\right]\cdot\left\{\int_0^{1/4} g(t)dt\right\}$$

$$+W(\phi+\Gamma_4)\left[\int_{3/4}^{1} dt\right]\cdot\left\{\int_{0}^{1/4} g(t)dt\right\}$$

$$+(\phi+\Gamma_1)\cdot\left\{\int_{0}^{1/4}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]ds\right\}\cdot\left\{\int_{1/4}^{1/2} g(t)dt\right\}+(\phi+\Gamma_2)\cdot\left\{\int_{1/4}^{\bar{t}_2}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]ds\right\}\cdot\left\{\int_{1/4}^{1/2} g(t)dt\right\}$$

$$+\ W(\phi+\Gamma_2)\left[\int_{0}^{1/2} dt-\int_{1/4}^{1/2} t\,\frac{g(t)}{\int_{1/4}^{1/2} g(t)dt}\,dt\right]\cdot\left\{\int_{1/4}^{1/2} g(t)dt\right\}+W(\phi+\Gamma_3)\left[\int_{1/2}^{3/4} dt\right]\cdot\left\{\int_{1/4}^{1/2} g(t)dt\right\}$$

$$+\ W(\phi+\Gamma_4)\left[\int_{3/4}^{1} dt\right]\cdot\left\{\int_{1/4}^{1/2} g(t)dt\right\}$$

$$+(\phi+\Gamma_1)\cdot\left\{\int_{0}^{1/4}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]ds\right\}\cdot\left\{\int_{1/2}^{3/4} g(t)dt\right\}+(\phi+\Gamma_2)\cdot\left\{\int_{1/4}^{1/2}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]ds\right\}\cdot\left\{\int_{1/2}^{3/4} g(t)dt\right\}$$

$$+(\phi+\Gamma_3)\left\{\int_{1/2}^{\bar{t}_3}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc]\ ds\right]\right\}\left\{\int_{1/2}^{3/4} g(t)dt\right\}+W(\phi+\Gamma_3)\left[\int_{0}^{3/4} dt-\int_{1/2}^{3/4} t\,\frac{g(t)}{\int_{1/2}^{3/4} g(t)dt}\,dt\right]\left\{\int_{1/2}^{3/4} g(t)dt\right\}$$

$$+W(\phi+\Gamma_4)\cdot\left[\int_{3/4}^{1} dt\right]\cdot\left\{\int_{1/2}^{3/4} g(t)dt\right\}$$

$$+(\phi+\Gamma_1)\cdot\left\{\int_0^{1/4}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc]\;ds\right]\right\}\cdot\left\{\int_{3/4}^{1} g(t)dt\right\}+(\phi+\Gamma_2)\cdot\left\{\int_{1/4}^{1/2}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]ds\right\}\cdot\left\{\int_{3/4}^{1} g(t)dt\right\}$$

$$+(\phi+\Gamma_3)\left\{\int_{1/2}^{3/4}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]ds\right\}\left\{\int_{3/4}^{1} g(t)dt\right\}+(\phi+\Gamma_4)\left\{\int_{3/4}^{\bar{t}_4}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]ds\right\}\left\{\int_{3/4}^{1} g(t)dt\right\}$$

$$+\;W(\phi+\Gamma_4)\cdot\left[\int_0^1 dt-\int_{3/4}^{1} t\,\frac{g(t)}{\int_{3/4}^{1} g(t)dt}\,dt\right]\left\{\int_{3/4}^{1} g(t)dt\right\}$$

$$+\left\{\begin{array}{l}(\phi+\Gamma_1)\cdot\int_0^{1/4}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)dc\right]ds+(\phi+\Gamma_2)\cdot\int_{1/4}^{1/2}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)dc\right]ds\\ \qquad+(\phi+\Gamma_3)\cdot\int_{1/2}^{3/4}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)dc\right]ds+(\phi+\Gamma_4)\cdot\int_{3/4}^{1}\left[\int_{-\infty}^{\infty} cp(c_0,c;s)dc\right]ds\end{array}\right\}\left\{1-\int_0^{T_M} g(t)dt\right\}$$

$$+r_N\cdot\int_0^{T_M}\left\{W-\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]\right\}ds\cdot\left[1-\int_0^{T_M} g(t)dt\right]+r_N\cdot\int_0^{\bar{t}_W}\left\{W-\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]\right\}ds\cdot\int_0^{T_M} g(t)dt$$

$$+r_1\cdot\int_0^{\bar{t}_R}\left\{R-\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]\right\}ds\cdot\int_0^{T_M} g_R(t)dt+r_1\cdot\int_0^{T_M}\left\{R-\left[\int_{-\infty}^{\infty} cp(c_0,c;s)\,dc\right]\right\}ds\cdot\left[1-\int_0^{T_M} g_R(t)dt\right]$$

$$+ r_2 G \cdot \int_0^{T_M} dt \cdot \left[1 - \int_0^{T_M} g_R(t) dt \right]$$

$$+ \left\{ r_2 G \cdot \int_0^{\bar{t}_R} dt + r_1 \cdot \int_{\bar{t}_R}^{T_M} \left\{ R + G - \left[\int_{-\infty}^{\infty} cp(c_0, c; s)\, dc \right] \right\} ds \right\} \cdot \left[\int_0^{T_M} g_R(t) dt - \int_0^{T_M} g_{R+G}(t) dt \right]$$

$$+ r_2 G \cdot \int_0^{\bar{t}_{x_1}} dt \cdot \int_0^{T_M} g_{R+G}(t) dt + r_1 \cdot \int_{\bar{t}_R}^{\bar{t}_{R+G}} \left\{ R + G - \left[\int_{-\infty}^{\infty} cp(c_0 c; s) \right] dc \right\} ds \cdot \int_0^{T_M} g_{R+G}(t) dt$$

$$- \left\{ vc_G \left[\int_{-\infty}^{\infty} cp(c_0, c; T_M)\, dc - \int_{-\infty}^{\infty} cp(c_0, c; \bar{t}_R)\, dc \right] + FC_G \right\} \cdot \left[\int_0^{T_M} g_R(t) dt - \int_0^{T_M} g_{R+G}(t) dt \right]$$

$$- \left\{ vc_G \cdot \left[\int_{-\infty}^{\infty} cp(c_0, c; \bar{t}_{R+G})\, dc - \int_{-\infty}^{\infty} cp(c_0, c; \bar{t}_R)\, dc \right] + FC_2 \right\} \cdot \int_0^{T_M} g_{R+G}(t) dt$$

$$- \left\{ r_3 \cdot \int_{\bar{t}_{R+G}}^{\bar{t}_W} \left\{ \left[\int_{-\infty}^{\infty} cp(c_0, c; s)\, dc \right] - k \right\} ds + FC_B \right\} \cdot \int_0^{T_M} g(t) dt$$

$$-\left\{ r_3 \cdot \; W-(R+G) \; \cdot \left\{ T_M - \int_{\bar{t}_{R+G}}^{T_M} t \frac{g(t)}{\int_0^{T_M} g(t)dt} dt \right\} \cdot \int_0^{T_M} g(t)dt \right\}$$

$$-\left\{ r_3 \cdot \int_{\bar{t}_{R+G}}^{T_M} \left\{ \left[\int_{-\infty}^{\infty} cp(c_0, c; s)\, dc \right] - (R+G) \right\} ds + FC_B \right\} \cdot \left\{ 1 - \int_0^{T_M} g(t)dt \right\}.$$

$$-r_D D - r_k k$$

Scrutinizing the objective function, it is clear that the bank's decision variables have many points of incidence in E(π). The assets used to fund the credit line, R and G, are found in the expected profit equation 81 and 41 times, respectively. W, the negotiated size of the credit line, is found on the RHS of the profit equation more than 12 dozen times. The bank's mark-up ϕ explicitly appears in the objective function 32 times. Since W(≡W(γ, ϕ, Γ_i) is an artifact of ϕ, the impact of "W" is augmented by another 2 dozen appearance in E(π). While our final decision variable "k", bank capital, appears only once in E(π) it is found in the constraint that reconciles the LHS and RHS of the bank's balance sheet. The reader should keep in mind that W* becomes L as it is taken down. The loan size "L" at t_0 is considered to be term loans with an initial value L_0 that is given exogenously in this analysis.

Differentiating E(π), alternatively with respect to W, R, G and k, then setting each result equal to zero yields a system of integral differential equations. The four first order solutions are

lengthy and so confounded that they are barren of any intuition. Not only are the bank's decision variables found frequently in the optimality conditions but their appearances are highly disparate. For instance, W^* is in lower limits, upper limits and integrands of integrals not just once but dozens of times and not just in one of the FOCs but all four of them. These same observations are true of R^*, G^* and k^*. Not being able to isolate the decision variables on the left hand side of the first order conditions mutes any obvious insight. In addition, the complexity of the optimality conditions guarantees that the implicit function theorem will be fruitless in yielding any useful comparative static analysis. Consequently, we simulated the values of the bank's decision variables which satisfy the FOCs over a host of scenarios. In order to obtain solutions for W^*, R^* , G^* and k^*, we must select illustrative values for a host of parameters and variables. As has been clear since Section III, the length of the bank's planning period is one year. We chose r_1 = 2% and r_2 = 4% under the sole presumption that the rate of return on treasuries had to be greater than the rate that the Fed was paying on bank reserves shortly before the crisis. The 3 month $LIBOR_i\ (\equiv \Gamma_i)$ as chosen to be 6% across all four quarters. Clearly if the bank has definitive expectations about future $LIBOR_i$ rates then the intermediary could easily substitute those forecasts into $\Gamma_1, \Gamma_2, \Gamma_3,$ and Γ_4. The cost of external borrowing by the bank r_3 was set equal to 9%.

In regard to its fees on credit lines and the rate paid on the part of the credit line that has been taken down, the bank is a price maker and faces a downward sloping demand schedule. We recognized W as the demand for loan commitments by borrowers and write it as a function of the non-usage fee (r_N) and the rate paid on the takedown ($\Gamma_i + \phi$). An exponential functional form

was chosen for W(r_N , Γ_i , ϕ), though we treat both the non-usage fee r_N and Γ_i as being exogenous of decision makers they influence the demand for credit. The partial derivative of credit demanded with respect to ϕ is simply

$$\frac{\partial W(r_N, \Gamma_i, \phi)}{\partial \phi} = -\rho W(r_N, \Gamma_i, \phi)$$

detailing an inverse relationship between the magnitude of the loan commitment demand and the bank's fixed mark-up rate. We picked a value for ρ that would allow the elasticity of demand to be in the neighborhood of -1 (depending on the exact solutions for ϕ^* and W^*). We assumed that the bank would have to pay both the spread and a commission, so liquidating $100M of treasury securities could easily cost a half a million dollars. Consequently, we assigned "vc_G" the value of $.005 per dollar of treasuries converted to reserves.[21]

We used Newton's search technique to numerically solve for W^* , R^* , G^* and k^*. The optimality conditions for the bank's decision variates are given as

$$\begin{pmatrix} \dfrac{\partial E(\pi)}{\partial W^*(N+1)} \\ \dfrac{\partial E(\pi)}{\partial R^*(N+1)} \\ \dfrac{\partial E(\pi)}{\partial G^*(N+1)} \\ \dfrac{\partial E(\pi)}{\partial k^*(N+1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Using a first order Taylor expansion on the vector above yields

$$\begin{pmatrix} \frac{\partial E(\pi)}{\partial W^*(N)} \\ \frac{\partial E(\pi)}{\partial R^*(N)} \\ \frac{\partial E(\pi)}{\partial G^*(N)} \\ \frac{\partial E(\pi)}{\partial k^*(N)} \end{pmatrix} + \begin{bmatrix} \frac{\partial^2 E(\pi)}{\partial W^*(N)\partial W^*(N)} & \frac{\partial^2 E(\pi)}{\partial W^*(N)\partial R^*(N)} & \frac{\partial^2 E(\pi)}{\partial W^*(N)\partial G^*(N)} & \frac{\partial^2 E(\pi)}{\partial W^*(N)\partial k^*(N)} \\ \frac{\partial^2 E(\pi)}{\partial R^*(N)\partial W^*(N)} & \frac{\partial^2 E(\pi)}{\partial R^*(N)\partial R^*(N)} & \frac{\partial^2 E(\pi)}{\partial R^*(N)\partial G^*(N)} & \frac{\partial^2 E(\pi)}{\partial R^*(N)\partial k^*(N)} \\ \frac{\partial^2 E(\pi)}{\partial G^*(N)\partial W^*(N)} & \frac{\partial^2 E(\pi)}{\partial G^*(N)\partial R^*(N)} & \frac{\partial^2 E(\pi)}{\partial G^*(N)\partial G^*(N)} & \frac{\partial^2 E(\pi)}{\partial G^*(N)\partial k^*(N)} \\ \frac{\partial^2 E(\pi)}{\partial k^*(N)\partial W^*(N)} & \frac{\partial^2 E(\pi)}{\partial k^*(N)\partial R^*(N)} & \frac{\partial^2 E(\pi)}{\partial k^*(N)\partial G^*(N)} & \frac{\partial^2 E(\pi)}{\partial k^*(N)\partial k^*(N)} \end{bmatrix} \square$$

$$\begin{bmatrix} W^*(N+1) - W^*(N) \\ R^*(N+1) - R^*(N) \\ G^*(N+1) - G^*(N) \\ k^*(N+1) - k^*(N) \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{25}$$

Solving for $\begin{bmatrix} W^*(N+1) - W^*(N) \\ R^*(N+1) - R^*(N) \\ G^*(N+1) - G^*(N) \\ k^*(N+1) - k^*(N) \end{bmatrix}$, we get

$$\begin{pmatrix} \frac{\partial E(\pi)}{\partial W^*(N)} \\ \frac{\partial E(\pi)}{\partial R^*(N)} \\ \frac{\partial E(\pi)}{\partial G^*(N)} \\ \frac{\partial E(\pi)}{\partial k^*(N)} \end{pmatrix} = - \begin{bmatrix} \frac{\partial^2 E(\pi)}{\partial W^*(N)\partial W^*(N)} & \frac{\partial^2 E(\pi)}{\partial W^*(N)\partial R^*(N)} & \frac{\partial^2 E(\pi)}{\partial W^*(N)\partial G^*(N)} & \frac{\partial^2 E(\pi)}{\partial W^*(N)\partial k^*(N)} \\ \frac{\partial^2 E(\pi)}{\partial R^*(N)\partial W^*(N)} & \frac{\partial^2 E(\pi)}{\partial R^*(N)\partial R^*(N)} & \frac{\partial^2 E(\pi)}{\partial R^*(N)\partial G^*(N)} & \frac{\partial^2 E(\pi)}{\partial R^*(N)\partial k^*(N)} \\ \frac{\partial^2 E(\pi)}{\partial G^*(N)\partial W^*(N)} & \frac{\partial^2 E(\pi)}{\partial G^*(N)\partial R^*(N)} & \frac{\partial^2 E(\pi)}{\partial G^*(N)\partial G^*(N)} & \frac{\partial^2 E(\pi)}{\partial G^*(N)\partial k^*(N)} \\ \frac{\partial^2 E(\pi)}{\partial k^*(N)\partial W^*(N)} & \frac{\partial^2 E(\pi)}{\partial k^*(N)\partial R^*(N)} & \frac{\partial^2 E(\pi)}{\partial k^*(N)\partial G^*(N)} & \frac{\partial^2 E(\pi)}{\partial k^*(N)\partial k^*(N)} \end{bmatrix} \square$$

$$\begin{bmatrix} W^*(N+1) - W^*(N) \\ R^*(N+1) - R^*(N) \\ G^*(N+1) - G^*(N) \\ k^*(N+1) - k^*(N) \end{bmatrix}.$$

By multiplying both sides of the expression by the inverse of the Jacobian, we get

$$-\begin{bmatrix} \frac{\partial^2 E(\pi)}{\partial W^*(N)\partial W^*(N)} & \frac{\partial^2 E(\pi)}{\partial W^*(N)\partial R^*(N)} & \frac{\partial^2 E(\pi)}{\partial W^*(N)\partial G^*(N)} & \frac{\partial^2 E(\pi)}{\partial W^*(N)\partial k^*(N)} \\ \frac{\partial^2 E(\pi)}{\partial R^*(N)\partial W^*(N)} & \frac{\partial^2 E(\pi)}{\partial R^*(N)\partial R^*(N)} & \frac{\partial^2 E(\pi)}{\partial R^*(N)\partial G^*(N)} & \frac{\partial^2 E(\pi)}{\partial R^*(N)\partial k^*(N)} \\ \frac{\partial^2 E(\pi)}{\partial G^*(N)\partial W^*(N)} & \frac{\partial^2 E(\pi)}{\partial G^*(N)\partial R^*(N)} & \frac{\partial^2 E(\pi)}{\partial G^*(N)\partial G^*(N)} & \frac{\partial^2 E(\pi)}{\partial G^*(N)\partial k^*(N)} \\ \frac{\partial^2 E(\pi)}{\partial k^*(N)\partial W^*(N)} & \frac{\partial^2 E(\pi)}{\partial k^*(N)\partial R^*(N)} & \frac{\partial^2 E(\pi)}{\partial k^*(N)\partial G^*(N)} & \frac{\partial^2 E(\pi)}{\partial k^*(N)\partial k^*(N)} \end{bmatrix} \square \begin{bmatrix} \frac{\partial E(\pi)}{\partial W^*(N)} \\ \frac{\partial E(\pi)}{\partial R^*(N)} \\ \frac{\partial E(\pi)}{\partial G^*(N)} \\ \frac{\partial E(\pi)}{\partial k^*(N)} \end{bmatrix} =$$

$$\begin{bmatrix} W^*(N+1) \\ R^*(N+1) \\ G^*(N+1) \\ k^*(N+1) \end{bmatrix} - \begin{bmatrix} W^*(N) \\ R^*(N) \\ G^*(N) \\ k^*(N) \end{bmatrix}$$

Finally, we isolate the solutions for the bank's decision variables at iteration $N+1$ to obtain

$$\begin{bmatrix} W^*(N+1) \\ R^*(N+1) \\ G^*(N+1) \\ k^*(N+1) \end{bmatrix} = \begin{bmatrix} W^*(N) \\ R^*(N) \\ G^*(N) \\ k^*(N) \end{bmatrix}$$

$$-\begin{bmatrix} \frac{\partial^2 E(\pi)}{\partial W^*(N)\partial W^*(N)} & \frac{\partial^2 E(\pi)}{\partial W^*(N)\partial R^*(N)} & \frac{\partial^2 E(\pi)}{\partial W^*(N)\partial G^*(N)} & \frac{\partial^2 E(\pi)}{\partial W^*(N)\partial k^*(N)} \\ \frac{\partial^2 E(\pi)}{\partial R^*(N)\partial W^*(N)} & \frac{\partial^2 E(\pi)}{\partial R^*(N)\partial R^*(N)} & \frac{\partial^2 E(\pi)}{\partial R^*(N)\partial G^*(N)} & \frac{\partial^2 E(\pi)}{\partial R^*(N)\partial k^*(N)} \\ \frac{\partial^2 E(\pi)}{\partial G^*(N)\partial W^*(N)} & \frac{\partial^2 E(\pi)}{\partial G^*(N)\partial R^*(N)} & \frac{\partial^2 E(\pi)}{\partial G^*(N)\partial G^*(N)} & \frac{\partial^2 E(\pi)}{\partial G^*(N)\partial k^*(N)} \\ \frac{\partial^2 E(\pi)}{\partial k^*(N)\partial W^*(N)} & \frac{\partial^2 E(\pi)}{\partial k^*(N)\partial R^*(N)} & \frac{\partial^2 E(\pi)}{\partial k^*(N)\partial G^*(N)} & \frac{\partial^2 E(\pi)}{\partial k^*(N)\partial k^*(N)} \end{bmatrix} \square \begin{bmatrix} \frac{\partial E(\pi)}{\partial W^*(N)} \\ \frac{\partial E(\pi)}{\partial R^*(N)} \\ \frac{\partial E(\pi)}{\partial G^*(N)} \\ \frac{\partial E(\pi)}{\partial k^*(N)} \end{bmatrix}.$$

Of course, the magnitude of N depends upon the number of iterations it takes for the bank's optimal solutions to converge. A tolerance level of 10^{-7} was used in the simulations.[22]

5. Numeric Solutions to the FOCs

For the parameters specified in Section 4, the values of the decision variables that maximize the bank's intertemporal expected profit were obtained and appear in the second column of Table 1. In the base case, optimal reserves are \$3.020B, treasuries \$17.541B, loan commitments \$50.053B and bank capital \$6.118B. These solutions approximate the average size of identical accounts held by the 80 largest U.S. banks in the third quarter of 2007 and provide a familiar starting point for our analysis.

In order to gain a better understanding of W^*, R^*, G^* and k^*, as well as their interdependence, we perform a comparative static analysis upon the optimal solutions. The original parameter settings were alternatively increased by 10% and the simulated results are provided in the remaining six columns of the table. In the second set of simulations, we presume the Fed increases r_1 to 2.2%. The increase in asset profitability increases the scale of bank operations as capital rises by \$0.396B. The equity enhancement helps to finance an increase in credit line lending W^* from \$50.053B to \$50.327B. Intuitively, the increase in "IOR" augments the revenue associated with maintaining stored liquidity to finance uncertain loan takedowns. Consequently, lending is more profitable and, optimally, the bank does more of it. Bank reserves rise from \$3.020B to \$4.419B while optimal government security holdings fall by \$2.001B. Of

course, these results provide support for the Fed's decision to pay interest on reserves since it encourages both bank lending and capital formation.

In the third column comparative static results are detailed for an increase in r_2 to 4.4%. Again in this simulation the increase in asset profitability occasions an enhancement in k^*. The acquisition of capital provides additional permanent funding for a new higher level of G^* at \$22.075B while the bank reduces its optimal holdings of reserves. It is of note that W^* falls to \$48.9B as the intermediary substitutes security holdings for loan commitments. It follows from this analysis that the Fed's pursuance of QE1 and QE2 could very well be just the right line of attack during our current economic downturn. Quantitative Easing has essentially meant sustained efforts by the Fed to replace treasury securities with bank reserves. As a consequence of this behavior by the Board, the price of treasuries have risen dramatically and r_2 has fallen to historic lows. Of course, a reduction in the return to G is no more than the mirror reflection of this third simulation in Table 1. As such the fall in r_2 would, optimally, mean a fall in intermediary treasury holdings and an increase in credit line lending in an environment characterized by stochastic borrower takedowns.

The fourth set of simulation results investigates the impact of an increase in the cost of liquidating treasuries to fund the stochastic takedown of credit lines over time. The increase in "vc_G" reduces the attractiveness of the LHS of the bank's balance sheet and contracts the scale of operations as k^* retreats to \$5.982B. G^* falls from \$17.541B to \$16.216B since its

attractiveness as a source of stored liquidity is diminished. At the same time, the bank puts more reliance on reserves to provide immediate financing to an escalated level of W^*. The results found in column 6 record the new solutions to the first order conditions associated with a 10% increase in bank deposits. The change in deposits from $47.656B to $52.422B amounts to a $4.766B increase in permanent funding for LHS of the intermediary's tee account. Bank reliance upon equity for permanent funding falls from $6.118B to $5.276B since deposits are being used to provide substitute permanent funding for R^*, G^* and W^*. Credit line lending increases to $52.646B to take advantage of the increased deposit funding. Clearly, this simulation parallels the dramatic re-intermediation of bank transaction accounts that took place in September 2008. The results in Table 1 suggest that were it not for other forces the "flight to safety" by domestic depositors would have lead to increased credit line lending W^*. The sixth set of simulations detail just what happens to R^*, G^*, W^* and k^* if there is an increase in the elasticity of demand for loan commitments. The 10% increase in ρ reduces the attractiveness of credit line lending and occasions a reduction in k^* since W^* now needs less permanent financing. Part of the fall in W^* appears as an increase in the optimal holdings of government securities. The final set of simulations that appear in Table 1 documents the impact of a 10% decrease in the elasticity of the supply of bank capital. The contraction in the supply of equity schedule reduces k^* by 12.23% from its original base case solution of $6.118B. The deterioration in capital occasions a fall in R^*, G^* and W^* due to the decrease in permanent financing for the LHS of the intermediary's balance sheet.

In summary, the simulations investigate the relationship between R^*, G^*, W^* and k^* when loan timing and usage is recognized as being stochastic and the implications of this uncertainty is acknowledged. Alternative enhancements in returns offered by bank assets, r_1 then r_2, induce the bank to acquire more capital in both cases and occasions substitution among the competing assets. A surge in the source of permanent funding increases the optimal holdings of bank assets across the board. Conversely, a contraction in financing provided by the RHS of the intermediary's tee account reduces R^*, G^* and W^*. A strength of Table 1 must be just how intuitive the comparative static results are given the ambitious context of the simulations.

6. The Impact of Binding Capital Requirement Upon Optimal Bank Lending

Maximizing intertemporal expected profits with uncertain loan usage provides us with a realistic framework in which to investigate the relationship between capital requirements and intermediary lending. In particular, endogenizing equity allows us to speak to the conflict that has arisen in regard to the issue of adequate bank capital. While one school of thought maintains that equity has to rise in order to protect taxpayers from the next great financial crisis. Others argue that enacting binding capital requirements would discourage bank lending and stall the nation's recovery. Table 1 details the comparative static behavior of bank decision variables under nonbinding capital requirements (NB-CR) while Table 2 documents the same simulations when capital is constrained to be a fixed proportion of lending (B-CR). In Table 2, once again an increase in IOR makes assets more profitable and induces an increase in k^*. The increase in permanent funds supports an increase in R^* and W^*. Optimal treasury holdings increase with the

+.4% change in r_2 while both R^* and W^* fall since they become relatively less attractive to the bank. In the second table, an increase in the per dollar conversion cost of government securities reduces G^* from its base case solution and increases both R^* and W^*. A 10% increase in deposits supplied to the bank supports an increase in R^*, G^* and W^* since transactional accounts are a cheap source of permanent funding. An increase in the elasticity of demand for credit, in Table 2, reduces k^* and discourages both R^* and W^* while G^* increases relative to its base case FOC solution. In the final simulation provided by the second table, a decrease in the elasticity of supply of bank capital reduces k^* and W^*.

Clearly Table 2, in and of itself, is interesting but not very surprising since the comparative static results are intuitive and, generally, in the spirit of the results provided in the first table. However when Table 2 is compared and contrasted with Table 1, the impact of binding capital requirements becomes apparent. For example, consider an increase in r_2 to 4.4% then assets would become more productive seemingly encouraging bank capital under either regime. In Table 1, capital does indeed surge which funds an increase in G* while W* falls. Under binding capital requirements the enhanced attractiveness of treasuries discourages credit line lending. The fall in W* reduces k*. Given the current performance of the domestic economy let's re-interpret this result for a decrease in r_2. In particular, say the Fed buys treasuries to reduce the return to G* and support an economic recovery then, according to Table 2, both k* and W* will increase. Clearly this is a strong result that suggests expansionary monetary policy will reduce interest rates and encourage capital formation if equity requirements are binding. In contrast, under NB-CR, if the Fed reduces r_2 endogenous bank capital will fall.

In times of financial stress, it is often the case that many banks are simultaneously trying to liquidate assets and, consequently, conversion costs increase. An increase in "vc_G" induces the bank to reallocate its assets in favor of W^* at the expense of G^* under both capital requirement regimes (binding and non-binding). The LHS of the bank's balance sheet is not as profitable as it was before the hike in "vc_G", consequently, the intermediary contracts the permanent funding it acquires and k^* falls under NB-CR. In Table 2, the increase in asset conversion costs encourages W^* which dictates an increase in bank equity. This result provides more support for binding capital requirements since the financial events that occasion increases in "vc_G" should be greeted with increasing levels of bank equity.

An autonomous increase in deposits from $47.656B to $52.422B provides permanent funding for increases in R^*, G^* and W^* in both Table 1 and Table 2. In the case of binding equity requirements, the increase in credit line lending demands an increase in k^* as well. However, under NB-CR, the solution to the FOC for optimal capital actually falls since deposits have become a more attractive source of funding for the bank's assets. This result is disconcerting in the sense that exogenous increases in the supply of bank deposits are normally associated with "flights to quality." Funds often flow into transactional accounts during moments of economic and financial stress. These are times when regulatory authorities would prefer that banks have greater levels of capital on hand. However, our results suggest that an influx of funds into

depository accounts could optimally be used to reduce the bank's reliance on equity when capital requirements are not binding.

Under the two capital requirement schemes, the six alternative parameter changes move k^* in the same direction in the three cases but in opposite directions in the three remaining circumstances. Increases in asset conversion costs for a bank and in the level of intermediary transactional deposits are economic occurrences associated with stress in the financial sector. The fact that optimizing behavior, in the face of these events, dictates a reduction in unconstrained equity is reason alone to ensure that capital requirements are binding. Furthermore, the optimal level of credit line lending is greater under B-CR with either the increase in conversion costs or the increase in bank deposits. When parametric changes move capital in the same direction in Tables 1 and 2, the adjustments of W^* and k^* are smaller in magnitude under the B-CR regime. Realistically changes in r_2 or ρ or γ will impact all commercial banks simultaneously. The unconstrained recalibration of credit line lending and equity under NB-CR will generate more market volatility than will the change in these accounts under binding requirements. While the restrained adjustments of W^* and k^* will be less disruptive to financial markets, they also provide further support for the idea of effectively regulating the amount of equity a bank must hold.

7. The Expected Time to Liquidity Depletion

In the simulations provided by Tables 1 and 2 expected intertemporal profits were maximized by the bank under binding and nonbinding capital requirements. The determination of R^* and G^* was strictly on the basis of their respective contributions to $E(\pi)$. However, in regard to

safeguarding the financial system, the Federal Reserve is interested in the ability of individual banks to meet their liquidity needs when markets are "stressed". The financial crisis of 2008 occasioned just such a scenario. Not just one bank but many intermediaries simultaneously suffered sustained pressure on their reserves (primary and secondary) as deposits were liquidated, credit lines were taken down and interbank borrowing markets were frozen. In times like these, regulators would like banks to be able to satisfy a minimum liquidity coverage ratio LCR of 1 where:

_____ .

This metric ensures that the intermediary maintains an adequate level of "unencumbered, high quality assets" that can be used to meet its cash needs for at least 30 days. The Fed assumes that within a month's time that the bank, at hand, can take action to resolve the impact of market stress in an orderly way. The optimal level of bank liquidity, detailed in the simulations, has a likelihood of depletion given by RLS (risk of a liquidity shortage)

$$RLS = \int_0^{T_M} [\frac{q_i}{\sqrt{2\pi\sigma^2 t^3}}\ e^{-(q_i-\mu t)^2/2\sigma^2 t}]dt$$

and, consequently, an expected time to depletion of q_i which is given by

$$E_{q_i}[T] = \int_0^{T_M} t\ [\frac{q_i}{\sqrt{2\pi\sigma^2 t^3}}\ e^{-(q_i-\mu t)^2/2\sigma^2 t}]dt$$

or

$$E_{q_i}[T] = \int_0^{T_M} t\ g_{q_i}(t)dt$$

for q_i, alternatively, equal to R^* and $(R^* + G^*)$.

The results recorded in Tables 3 and 4 should afford regulators a great deal of comfort. If high quality liquid assets are strictly interpreted as bank reserves then regardless of the impact of k^* upon R^* the bank is always left with at least 21 days of protection. Whether capital requirements are binding or nonbinding the shortest expected time to depletion of bank reserves is calculated to be 20.98 days. Alternatively, in Table 4, if "unencumbered, high quality assets" is taken to mean the bank's reserves plus its treasury holdings then the intermediary should expect to have at least 143.47 days before depository liquidity is exhausted. This time to expiry is shortest horizon recorded under either the NB-CR regime or the B-CR scheme. The reader should keep in mind that the determination of R^* and of G^* are both under the assumption that the bank maximizes its intertemporal profits not that it satisfies the Federal Reserve's liquidity coverage ratio. Despite this, the average LCR for $(R^* + G^*)$ is just about 5 times as large as the expectations of regulators.

8. Summary

The intent of this research was to characterize optimal bank capital when stochastic credit line usage was acknowledged. We proxied uncertain loan takedown with trended Brownian motion. Relying on "time to first passage" mathematics, we derived a probability density function for the time to depletion of the bank credit line as well as the likelihoods for the time to exhausting the sources of liquidity that fund the loan takedown. Armed with these analytical results, we solved for the level of bank capital as well as reserves, treasuries, and the size of credit lines that maximize the intermediary's expected intertemporal profits. The optimality conditions produced a system of integral differential equations which refused to yield reduced form solutions and provided very little intuition. Consequently, the bank decision variables which satisfied the FOCs were simulated over a host of realistic scenarios. Under NB-CR, alternative enhancements in the returns offered by reserves and treasuries induced the bank to acquire more capital in both cases. While the increase in r_1 and then r_2 occasioned predictable substitutions among the competing assets. A surge in the source of permanent funding increased the optimal holdings of bank assets across the board. Conversely, a contraction in the supply of capital schedule reduced R^*, G^* and W^*.

The results recorded for optimal bank behavior in Table 2, under binding capital requirements, were very much in the spirit of those provided in the first table. However, there were some salient differences that support the idea of regulating intermediary equity. An increase in either asset conversion costs or in the level of transactional deposit balances are occurrences associated with stress in the financial sector. Our simulations recorded a reduction in optimal capital under NB-CR in the face of each of these events. While when equity is constrained, k^* increases.

These two results alone are convincing support for binding capital requirements. Furthermore, the third set of simulations in Table 2 establish that if the Fed were to pursue expansionary monetary policy and reduce r_2 then both W^* and k^* would increase. While, under NB-CR, Fed purchases of treasuries reduce r_2 and encourage credit line commitments but optimal intermediary capital falls. The increase in bank equity during periods of expansionary open market operations, in the second table, provides additional support for the idea of binding capital requirements.

Table 1: Comparative Static Behavior of Bank Decision Variables With Stochastic Takedown of the Credit Line with NB-CR (B=billions)

Parameter	Base Case Value	Given a 10% Increase in r_1	Given a 10% Increase in r_2	Given a 10% Increase in vc_G	Given a 10% Increase in D	Given a 10% Increase in ρ	Given a 10% Decrease in γ
r_1	2%	2.2%	2 %	2%	2%	2%	2%
r_2	4%	4 %	4.4%	4%	4%	4%	4%
vc_G	$.005/$	$.005/$	$.005/$	$.0055/$	$.005/$	$.005/$	$.005/$
D	$47.656B	$47.656B	$47.656B	$47.656B	$52.422B	$47.656B	$47.656B
ρ	40	40	40	40	40	44	40
γ	12.5	12.5	12.5	12.5	12.5	12.5	13.75
W^*	$50.053B	$50.327B	$48.900B	$50.198B	$52.646B	$48.009B	$49.547B
R^*	$ 3.020B	$ 4.419B	$ 2.877B	$ 3.714B	$ 3.914B	$ 2.933B	$ 2.991B
G^*	$17.541B	$15.536B	$22.075B	$16.216B	$18.648B	$17.893B	$17.329B
k^*	$ 6.118B	$ 6.514B	$ 6.971B	$ 5.982B	$ 5.276B	$ 5.274B	$ 5.370B

Table 2: Comparative Static Behavior of Bank Decision Variables With Stochastic Takedown of the Credit Line with B-CR (B=billions)

Parameter	Base Case Value	Given a 10% Increase in r_1	Given a 10% Increase in r_2	Given a 10% Increase in vc_G	Given a 10% Increase in D	Given a 10% Increase in ρ	Given a 10% Decrease in γ
r_1	2%	2.2%	2 %	2%	2%	2%	2%
r_2	4%	4%	4.4%	4%	4%	4%	4%
vc_G	$.005/$	$.005/$	$.005/$	$.0055/$	$.005/$	$.005/$	$.005/$
D	$47.656B	$47.656B	$47.656B	$47.656B	$52.422B	$47.656B	$47.656B
ρ	40	40	40	40	40	44	40
γ	12.5	12.5	12.5	12.5	12.5	12.5	13.75
W^*	$50.053B	$50.149B	$48.380B	$50.281B	$53.248B	$48.315B	$49.924B
R^*	$ 3.020B	$ 4.299B	$ 2.947B	$ 3.527B	$ 3.771B	$ 2.979B	$ 3.014B
G^*	$17.541B	$15.409B	$21.649B	$16.272B	$19.053B	$18.122B	$17.582B
k^*	$ 6.118B	$ 6.129B	$ 5.913B	$ 6.146B	$ 6.508B	$ 5.906B	$ 6.102B

Table 3: Expected Time to Depletion of Bank Liquidity
With Stochastic Takedown under NB-CR
(B=billions, d=days)

Parameter	Base Case Value	Given a 10% Increase in r_1	Given a 10% Increase in r_2	Given a 10% Increase in vc_G	Given a 10% Increase in D	Given a 10% Increase in ρ	Given a 10% Decrease in γ
k^*	\$6.118B	\$6.514B	\$6.971B	\$5.982B	\$5.276B	\$5.274B	\$5.370B
R^*	\$3.020B	\$4.419B	\$2.877B	\$3.714B	\$3.914B	\$2.933B	\$2.991B
$E_{R^*}[T]$	22.02d	32.22d	20.98d	27.08d	28.54d	21.39d	21.81d
$R^* + G^*$	\$20.561B	\$19.955B	\$24.952B	\$19.930B	\$22.562B	\$20.823B	\$20.321B
$E_{R^*+G^*}[T]$	149.58d	145.24d	180.45d	145.06d	163.83d	151.47d	147.87d

Table 4: Expected Time to Depletion of Bank Liquidity
With Stochastic Takedown under B-CR
(B=billions, d=days)

Parameter	Base Case Value	Given a 10% Increase in r_1	Given a 10% Increase in r_2	Given a 10% Increase in vc_G	Given a 10% Increase in D	Given a 10% Increase in ρ	Given a 10% Decrease in γ
k^*	\$6.118B	\$6.129B	\$5.913B	\$6.146B	\$6.508B	\$5.906B	\$6.102B
R^*	\$3.020B	\$4.299B	\$2.947B	\$3.527B	\$3.771B	\$2.979B	\$3.014B
$E_{R^*}[T]$	22.02d	31.35d	21.49d	25.72d	27.50d	21.72d	21.98d
$R^* + G^*$	\$20.561B	\$19.709B	\$24.596B	\$19.797B	\$22.821B	\$21.102B	\$20.590B
$E_{R^*+G^*}[T]$	149.58d	143.47d	178.01d	144.10d	165.66d	153.46d	149.80d

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Appendix A

Suppose that the random variable $C(t)$ representing the process in question takes the value c_0 at time t_0 and c_1 at time t_1, i.e., $C(t_0) = c_0$ and $C(t_1) = c_1$. Let me write $p(c_0, t_0; c_1, t_1)$ for the conditional frequency function of the variable c_1 at time t_1 given the value c_0 at the previous time t_0. Notice that the order of the pairs (c_0, t_0) and (c_1, t_1) represent the direction of the transition. I now consider three epochs of time $t_0 < t_1 < t_2$, with the corresponding variable values c_0, c_1, and c_2 where

$$p(c_0, t_0; c_2, t_2) = \int_{-\infty}^{\infty} p(c_0, t_0; c_1, t_1)\, p(c_1, t_1; c_2, t_2)\, dc_1. \qquad \text{(i)}$$

I am assuming the usual Markov property that, given the present state of the system, the future behavior does not depend on the past. A proof of the equation above can be provided by considering first a path from (c_0, t_0) to (c_2, t_2) through a particular intermediate point (c_1, t_1). The probability of this specific path for a Markov process is $p(c_0, t_0; c_1, t_1)\, p(c_1, t_1; c_2, t_2)$. Thus the total probability for a transition from (c_0, t_0) to (c_2, t_2) is obtained by integrating over all possible intermediate points, i.e., integrating with respect to c_1 as shown above.

Suppose I consider, for $t_0 < t_1$, the difference

$$p(c_0, t_0 - \Delta t; c_1, t_1) - p(c_0, t_0; c_1, t_1). \qquad \text{(ii)}$$

From the expression (i) above, I have

$$p(c_0, t_0 - \Delta t; c_1, t_1) = \int_{-\infty}^{\infty} p(c_0, t_0 - \Delta t; z, t_0)\, p(z, t_0; c_1, t_1)\, dz\ .$$

Furthermore, I can write

$$p(c_0,t_0;c_1,t_1) = p(c_0,t_0;c_1,t_1)\int_{-\infty}^{\infty} p(c_0,t_0-\Delta t_0;z,t_0)dz$$

$$\text{because clearly } \int_{-\infty}^{\infty} p(c_0,t_0-\Delta t_0;z,t_0)dz = 1.$$

From here, I have

$$p(c_0,t_0-\Delta t_0;c_1,t_1) - p(c_0,t_0;c_1,t_1) = \int_{-\infty}^{\infty} p(c_0,t_0-\Delta t_0;z,t_0)[p(z,t_0;c_1,t_1) - p(c_0,t_0;c_1,t_1)]dz. \quad \text{(iii)}$$

Rewriting the second term in the integrand in terms of a Taylor expansion, yields

$$\int_{-\infty}^{\infty} p(c_0,t_0-\Delta t_0;z,t_0)[(z-c_0)\frac{\partial p(c_0,t_0;c_1,t_1)}{\partial c_0} + \frac{1}{2}(z-c_0)^2\frac{\partial^2 p(c_0,t_0;c_1,t_1)}{\partial c_0 \partial c_0}]dz\,.$$

So that I have

$$p(c_0,t_0-\Delta t_0;c_1,t_1) - p(c_0,t_0;c_1,t_1) = \frac{\partial p(c_0,t_0;c_1,t_1)}{\partial c_0}\int_{-\infty}^{\infty}(z-c_0)p(c_0,t_0-\Delta t_0;z,t_0)dz \quad \text{(iv)}$$
$$+\frac{1}{2}\frac{\partial^2 p(c_0,t_0;c_1,t_1)}{\partial c_0 \partial c_0}\int_{-\infty}^{\infty}(z-c_0)^2 p(c_0,t_0-\Delta t_0;z,t_0)dz.$$

Let me now suppose that there exist infinitesimal means and variances for changes in the

basic random variable $C(t)$ defined by

$$\mu(c_0,t_0)=\lim_{\Delta t_0\to 0}\frac{1}{\Delta t_0}\int_{-\infty}^{\infty}(z-c_0)\,p(c_0,t_0-\Delta t_0;z,t_0)dz \tag{v}$$

$$\sigma^2(c_0,t_0)=\lim_{\Delta t_0\to 0}\frac{1}{\Delta t_0}\int_{-\infty}^{\infty}(z-c_0)^2\,p(c_0,t_0-\Delta t_0;z,t_0)dz\ .$$

I may have to restrict the ranges of integration on the right hand side of (v) to ensure convergence, but it is unnecessary to pursue this point in my present non-rigorous discussion.

Finally, dividing both sides of (iv) by Δt and letting $\Delta t\to 0$ yields:

$$-[\frac{p(c_0,t_0;c_1,t_1)-p(c_0,t_0-\Delta t_0;c_1,t_1)}{\Delta t_0}]=\frac{\partial p(c_0,t_0;c_1,t_1)}{\partial c_0}\mu(c_0,t_0)+\frac{1}{2}\frac{\partial^2 p(c_0,t_0;c_1,t_1)}{\partial c_0\partial c_0}\sigma^2(c_0,t_0)$$

or

$$-[\frac{\partial p(c_0,t_0;c_1,t_1)}{\partial t_0}]=\mu(c_0,t_0)\frac{\partial p(c_0,t_0;c_1,t_1)}{\partial c_0}+\frac{1}{2}\sigma^2(c_0,t_0)\frac{\partial^2 p(c_0,t_0;c_1,t_1)}{\partial c_0\partial c_0}.$$

Writing $p(c_0,c;t)$ as the probability density function of c at time t for the time invariant case, I have

$$-[\frac{\partial p(c_0,c;t)}{\partial t_0}]=\mu\frac{\partial p(c_0,c;t)}{\partial c_0}+\frac{1}{2}\sigma^2\frac{\partial^2 p(c_0,c;t)}{\partial c_0\partial c_0} \tag{vi}$$

This then is the backward Kolmogorov diffusion equation for a continuous time stochastic process.

Appendix B

Consider $p(c_0, c; t) = \frac{1}{\sigma\sqrt{2\pi t}} e^{\frac{-(c\ -c_0-\mu t)^2}{2\sigma^2 t}}$

$$\frac{\partial p(\cdot)}{\partial t} = -\frac{1}{2}\frac{t^{-3/2}}{\sigma\sqrt{2\pi}} e^{\frac{-(c-c_0-\mu t)^2}{2\sigma^2 t}} + \frac{1}{\sigma\sqrt{2\pi t}} e^{\frac{-(c-c_0-\mu t)^2}{2\sigma^2 t}} [\frac{2(c-c_0-\mu t)\mu[2\sigma^2 t] + 2\sigma^2(c-c_0-\mu t)^2}{4\sigma^4 t^2}]$$

$$\frac{\partial p(\cdot)}{\partial c_0} = p(c_0, c; t)\frac{-2(c-c_0-\mu t)}{2\sigma^2 t}(-1)$$

$$\frac{\partial^2 p(\cdot)}{\partial c_0 \partial c_0} = p(c_0, c; t)\frac{(c-c_0-\mu t)^2}{\sigma^4 t^2} + p(c_0, c; t)\frac{-1}{\sigma^2 t}$$

Substituting into the backward Kolmogorov equation yields (since t_0 has been suppressed in my notation I use t where $\frac{\partial p\ \cdot}{\partial t} = (-1)\frac{\partial p\ \cdot}{\partial t_0}$)

$$-\frac{1}{2}t^{-1}p(\cdot) + p(\cdot)[\frac{4(c-c_0-\mu t)\mu\sigma^2 t + 2\sigma^2(c-c_0-\mu t)^2}{4\sigma^4 t^2}]$$

$$= p(\cdot)\frac{(c-c_0-\mu t)\mu}{\sigma^2 t} + \frac{1}{2}p(\cdot)[\frac{(c-c_0-\mu t)^2}{\sigma^4 t^2} - \frac{1}{\sigma^2 t}]\sigma^2$$

$$-\frac{1}{2}t^{-1} + [\frac{(c-c_0-\mu t)\mu t + \frac{1}{2}(c-c_0-\mu t)^2}{\sigma^2 t^2}] = \frac{(c-c_0-\mu t)\mu}{\sigma^2 t} + \frac{(c-c_0-\mu t)^2}{2\sigma^2 t^2} - \frac{1}{2t}$$

$$\frac{t(c-c_0-\mu t)\mu + \frac{1}{2}(c-c_0-\mu t)^2}{\sigma^2 t^2} = \frac{(c-c_0-\mu t)\mu}{\sigma^2 t} + \frac{(c-c_0-\mu t)^2}{2\sigma^2 t^2}$$

$$\frac{(c-c_0-\mu t)^2}{2\sigma^2 t^2} = \frac{(c-c_0-\mu t)^2}{2\sigma^2 t^2},$$

so $p(\cdot) = \dfrac{1}{\sigma\sqrt{2\pi t}} e^{\frac{-(c-c_0-\mu t)^2}{2\sigma^2 t}}$ satisfies the Kolmogorov diffusion equation.

Appendix C

$$\frac{\partial P(c_0, x_1; t)}{\partial t} = \mu \frac{\partial P(c_0, x_1; t)}{\partial c_0} + \frac{1}{2}\sigma^2 \frac{\partial^2 P(c_0, x_1; t)}{\partial c_0 \partial c_0}$$

because if

$$-\frac{\partial p(c_0, c; t)}{\partial t_0} = \mu \frac{\partial p(c_0, c; t)}{\partial c_0} + \frac{1}{2}\sigma^2 \frac{\partial p(c_0, c; t)}{\partial c_0 \partial c_0}$$

then

$$\frac{\partial p(c_0, c; t)}{\partial t} = \mu \frac{\partial p(c_0, c; t)}{\partial c_0} + \frac{1}{2}\sigma^2 \frac{\partial p(c_0, c; t)}{\partial c_0 \partial c_0}$$

then

$$\frac{\partial \int_{-\infty}^{x_1} p(c_0, c; t)\, dc}{\partial t} = \mu \frac{\partial \int_{-\infty}^{x_1} p(c_0, c; t)\, dc}{\partial c_0} + \frac{1}{2}\sigma^2 \frac{\partial \int_{-\infty}^{x_1} p(c_0, c; t)\, dc}{\partial c_0 \partial c_0}$$

$$\frac{\partial P(c_0, x_1; t)}{\partial t} = \mu \frac{\partial P(c_0, x_1; t)}{\partial c_0} + \frac{1}{2}\sigma^2 \frac{\partial^2 P(c_0, x_1; t)}{\partial c_0 \partial c_0}.$$

Appendix D

I use $-[\frac{\partial g(t;c_0,W)}{\partial t}]$ instead of $\frac{\partial g(t;c_0,W)}{\partial t_0}$ because I suppressed t_0 in the notation.

$$\frac{\partial g}{\partial t} = \mu \frac{\partial g}{\partial c_0} + \frac{1}{2}\sigma^2 \frac{\partial^2 g}{\partial c_0 \partial c_0}$$

$$g = \frac{W - c_0}{\sigma\sqrt{2\pi t^3}} e^{\frac{-(W-c_0-\mu t)^2}{2\sigma^2 t}} = \frac{t^{-3/2}(W - c_0)}{\sigma\sqrt{2\pi}} e^{\frac{-(W-c_0-\mu t)^2}{2\sigma^2 t}}$$

$$\frac{\partial g}{\partial t} = \frac{-(\frac{3}{2})t^{-5/2}(W - c_0)}{\sigma\sqrt{2\pi}} e^{\frac{-(W-c_0-\mu t)^2}{2\sigma^2 t}} + g[\frac{2\mu(W - c_0 - \mu t)(2\sigma^2 t) + (W - c_0 - \mu t)^2 2\sigma^2}{4\sigma^2\sigma^2 t^2}]$$

$$\frac{\partial g}{\partial c_0} = \frac{-1}{\sigma\sqrt{2\pi t^3}} e^{\frac{-(W-c_0-\mu t)^2}{2\sigma^2 t}} + \frac{(W - c_0)}{\sigma\sqrt{2\pi t^3}} e^{\frac{-(W-c_0-\mu t)^2}{2\sigma^2 t}} [\frac{W - c_0 - \mu t}{\sigma^2 t}]$$

$$\frac{\partial^2 g}{\partial c_0 \partial c_0} = \frac{-1}{\sigma\sqrt{2\pi t^3}} e^{\frac{-(W-c_0-\mu t)^2}{2\sigma^2 t}} \cdot [\frac{W - c_0 - \mu t}{\sigma^2 t}] + \frac{-1}{\sigma\sqrt{2\pi t^3}} e^{\frac{-(W-c_0-\mu t)^2}{2\sigma^2 t}} \cdot [\frac{W - c_0 - \mu t}{\sigma^2 t}]$$

$$+ \frac{W - c_0}{\sigma\sqrt{2\pi t^3}} e^{\frac{-(W-c_0-\mu t)^2}{2\sigma^2 t}} [\frac{(W - c_0 - \mu t)^2}{\sigma^2\sigma^2 t^2}] + \frac{(W - c_0)}{\sigma\sqrt{2\pi t^3}} e^{\frac{-(W-c_0-\mu t)^2}{2\sigma^2 t}} [\frac{-1}{\sigma^2 t}]$$

I can see that Kolmogorov's diffusion equation holds because

$$\frac{\partial g}{\partial t} = -(\frac{3}{2})\frac{(W - c_0)}{\sigma\sqrt{2\pi} t^{\frac{5}{2}}} e^{(\cdot)} + g[\frac{\mu(W - c_0 - \mu t)}{\sigma^2 t}] + g[\frac{(W - c_0 - \mu t)^2}{2\sigma^2 t^2}]$$

$$\mu \frac{\partial g}{\partial c_0} = -[\frac{\mu}{\sigma\sqrt{2\pi t^3}} e^{(\cdot)}] + \mu \frac{(W - c_0)}{\sigma\sqrt{2\pi t^3}} e^{(\cdot)} [\frac{W - c_0 - \mu t}{\sigma^2 t}]$$

$$\frac{1}{2}\sigma^2 \frac{\partial^2 g}{\partial c_0 \partial c_0} = \frac{-1}{\sigma\sqrt{2\pi t^3}} e^{(\cdot)} \cdot [\frac{W - c_0 - \mu t}{t}] + \frac{1}{2}\frac{W - c_0}{\sigma\sqrt{2\pi t^3}} e^{(\cdot)} [\frac{(W - c_0 - \mu t)^2}{\sigma^2 t^2}] - \frac{1}{2}\frac{(W - c_0)}{\sigma\sqrt{2\pi t^3}} e^{(\cdot)} [\frac{1}{t}].$$

Plugging the right hand side of the three expressions above into the diffusion equation yields:

$$-(\frac{3}{2})\frac{(W-c_0)}{\sigma\sqrt{2\pi}t^{\frac{5}{2}}}e^{(\cdot)}+g[\frac{\mu(W-c_0-\mu t)}{\sigma^2 t}]+g[\frac{(W-c_0-\mu t)^2}{2\sigma^2 t^2}]=$$

$$-[\frac{\mu}{\sigma\sqrt{2\pi t^3}}e^{(\cdot)}]+\mu\frac{(W-c_0)}{\sigma\sqrt{2\pi t^3}}e^{(\cdot)}[\frac{W-c_0-\mu t}{\sigma^2 t}]$$

$$-[\frac{1}{\sigma\sqrt{2\pi t^3}}e^{(\cdot)}]\cdot[\frac{W-c_0-\mu t}{t}]+\frac{1}{2}g[\frac{(W-c_0-\mu t)^2}{\sigma^2 t^2}]-\frac{1}{2}\frac{(W-c_0)}{\sigma\sqrt{2\pi t^3}}e^{(\cdot)}[\frac{1}{t}].$$

So then

$$-(\frac{3}{2})\frac{(W-c_0)}{\sigma\sqrt{2\pi}t^{\frac{5}{2}}}e^{(\cdot)}+g[\frac{\mu(W-c_0-\mu t)}{\sigma^2 t}]=-[\frac{\mu}{\sigma\sqrt{2\pi t^3}}e^{(\cdot)}]+\mu\frac{(W-c_0)}{\sigma\sqrt{2\pi t^3}}e^{(\cdot)}[\frac{W-c_0-\mu t}{\sigma^2 t}]$$

$$-[\frac{1}{\sigma\sqrt{2\pi t^3}}e^{(\cdot)}]\cdot[\frac{W-c_0-\mu t}{t}]-\frac{1}{2}\frac{(W-c_0)}{\sigma\sqrt{2\pi t^3}}e^{(\cdot)}[\frac{1}{t}],$$

then

$$-(\frac{3}{2})\frac{(W-c_0)}{\sigma\sqrt{2\pi}t^{\frac{5}{2}}}e^{(\cdot)}+g[\frac{\mu(W-c_0-\mu t)}{\sigma^2 t}]=-[\frac{\mu}{\sigma\sqrt{2\pi t^3}}e^{(\cdot)}]+\mu g[\frac{W-c_0-\mu t}{\sigma^2 t}]$$

$$-[\frac{1}{\sigma\sqrt{2\pi t^3}}e^{(\cdot)}]\cdot[\frac{W-c_0-\mu t}{t}]-\frac{1}{2}\frac{(W-c_0)}{\sigma\sqrt{2\pi t^3}}e^{(\cdot)}[\frac{1}{t}]$$

and then

$$-[\frac{(W-c_0)}{\sigma\sqrt{2\pi}t^{\frac{5}{2}}}e^{(\cdot)}]+g[\frac{\mu(W-c_0-\mu t)}{\sigma^2 t}]=-[\frac{\mu}{\sigma\sqrt{2\pi t^3}}e^{(\cdot)}]+\mu g[\frac{W-c_0-\mu t}{\sigma^2 t}]-[\frac{1}{\sigma\sqrt{2\pi t^3}}e^{(\cdot)}]\cdot[\frac{W-c_0-\mu t}{t}].$$

After further cancellations I have

$$-[\frac{(W-c_0)}{\sigma\sqrt{2\pi}t^{\frac{5}{2}}}e^{(\cdot)}]=-[\frac{\mu}{\sigma\sqrt{2\pi t^3}}e^{(\cdot)}]-[\frac{1}{\sigma\sqrt{2\pi t^3}}e^{(\cdot)}]\cdot[\frac{W-c_0-\mu t}{t}].$$

And finally

$$-[\frac{(W-c_0)}{\sigma\sqrt{2\pi}t^{\frac{5}{2}}}]e^{(\frac{-(W-c_0-\mu t)^2}{2\sigma^2 t})}=-[\frac{(W-c_0)}{\sigma\sqrt{2\pi}t^{\frac{5}{2}}}]e^{(\frac{-(W-c_0-\mu t)^2}{2\sigma^2 t})}.$$

So I have shown the *pdf* for the time to a bank credit line being completely exhausted satisfies the Kolmogorov diffusion equation.

Appendix E

The Laplace transform is a method for solving differential equations. For example, consider

$$\frac{\partial f(x,t)}{\partial x} + x\frac{\partial f(\cdot)}{\partial t} = 0.$$

A Laplace transform of a linear combination of two functions is a linear combination of the Laplace transforms of each function and because a Laplace transform of zero is zero, I have

$$\mathrm{L}[\frac{\partial f(x,t)}{\partial x} + x\frac{\partial f(\cdot)}{\partial t}] = \mathrm{L}(0)$$

$$\mathrm{L}[\frac{\partial f(x,t)}{\partial x}] + \mathrm{L}[x\frac{\partial f(\cdot)}{\partial t}] = 0$$

$$\mathrm{L}[\frac{\partial f(x,t)}{\partial x}] + x\mathrm{L}[\frac{\partial f(\cdot)}{\partial t}] = 0\ .$$

I then change the order of the operators on the two terms appearing on the left hand side of the expression above

$$\frac{\partial \mathrm{L}(f)}{\partial x} + x[s\mathrm{L}(f) - f(x,0)] = 0$$

$$\frac{\partial \mathrm{L}(f)}{\partial x} + xs\mathrm{L}(f) = 0.$$

This then I regard as an ordinary differential equation in x since derivatives with respect to time do not occur in the equation.

If $\Omega = \mathrm{L}(f)$ then I have

$$\frac{\partial \Omega}{\partial x} + xs\Omega = 0 \text{ or I have}$$

$$\Omega(x,s) = \Omega_0 e^{\frac{-1}{2}sx^2}.$$

From here I have $\Omega(x,s) = \mathrm{L}(f)$, but to get $f(\cdot)$ I need to eventually obtain the solution for the Laplace inverse.

Detailing the fourth equation above, by definition $\mathrm{L}(f') = \int_0^\infty e^{-st} f'(t)dt$.

Using integration by parts

$$d(uv) = u\ dv + v\ du$$

$$uv\,|_b^a - \int_b^a u\ dv = \int_b^a v\ du$$

suppose that

$$v = e^{-st} \quad du = f'(t)dt\,.$$

So that I have

$$dv = -se^{-st}dt$$
$$u = f(t).$$

Plugging into our formula

$$f(t)e^{-st}\,|_b^a + s\int_b^a f(t)e^{-st}dt$$
$$f(t)e^{-st}\,|_0^\infty + s\int_0^\infty f(t)e^{-st}dt$$
$$-f(t=0)e^0 + s\mathrm{L}(f(t)) = \mathrm{L}(f'(t)).$$

Finally, I have

$$\mathrm{L}(f'(t)) = s\mathrm{L}(f(t)) - f(t=0)\,.$$

Consider a Laplace transform

$$\mathrm{L}[f(t);\rho] = \int_0^\infty e^{-\rho t} f(t)dt$$

where ρ is a positive constant such that $\mathrm{L}[f(t);\rho] = \int_0^\infty e^{-\rho t} f(t)dt$ converges.

Consider an example:

$$\mathrm{L}[1;\rho] = \int_0^\infty [1]e^{-\rho t}\, dt$$

$$-\frac{1}{\rho}e^{-\rho t}\,|_0^\infty = \frac{1}{\rho}.$$

Consider another example:

$$\mathrm{L}[t;\rho] = \int_0^\infty te^{-\rho t}dt$$

$$d(uv) = u\ dv + v\ du$$

$$uv\,|_b^a = \int_b^a v\ du + \int_b^a u\, dv$$

$$uv\,|_b^a - \int_b^a v\, du = \int_b^a u\, dv$$

$u = t$, $du = dt$, $dv = e^{-\rho t}dt$, and $v = -\frac{1}{\rho}e^{-\rho t}$

$$-\frac{1}{\rho}te^{-\rho t}\,|_0^\infty + \int_0^\infty \frac{1}{\rho}e^{-\rho t}dt$$

$$-\frac{1}{\rho^2}e^{-\rho t}\,|_0^\infty = \frac{1}{\rho^2}.$$

Appendix F

Consider

$$I(a,b)=\int_0^\infty e^{-a^2u^2-b^2u^{-2}}du\,.$$

If $v=au$, then I can rewrite the expression above as

$$\int_0^\infty e^{-v^2-b^2(\frac{a^2}{v^2})}du \text{ or as}$$

$$I(a,b)=\frac{1}{a}\int_0^\infty e^{-v^2-a^2b^2v^{-2}}dv \text{ using } v=au\,.$$

Again consider the original expression for *I(a,b)* and take its derivative with respect to b

$$\frac{\partial I(a,b)}{\partial b}=-2b\int_0^\infty u^{-2}e^{-a^2u^2-b^2u^{-2}}du\,.$$

Rewriting the derivative of $I(a,b)$, with respect to b, in terms of v

where this time $v=bu^{-1}$, I have $u^{-2}=(\frac{v}{b})^2$, $dv=-bu^{-2}du$, which yields

$$-2b\int_\infty^0(\frac{v}{b})^2e^{-a^2(\frac{b}{v})^2-b^2\frac{v^2}{b^2}}(-u^2b^{-1})dv.$$

Keep in mind when $u=0$ then $v=\infty$ and when $u=\infty$ then $v=0$, so I have

$$-2\int_0^\infty e^{-v^2-a^2b^2v^{-2}}dv\text{, so}$$

$$\frac{\partial I(a,b)}{\partial b} = -2\int_0^\infty e^{-v^2-a^2b^2v^{-2}}\,dv.$$

Recall from the top of page 13

$$I(a,b) = a^{-1}\int_0^\infty e^{-v^2-a^2b^2v^{-2}}\,dv$$

Consequently, I have

$$\frac{\dfrac{\partial I(a,b)}{\partial b}}{I(a,b)} = \frac{-2}{a^{-1}}$$

which implies $\dfrac{\partial I(a,b)}{\partial b} = -2aI(a,b)$.

The solution to the differential equation immediately above is $I(a,b) = I(a,0)e^{-2ab}$, where $I(a,0)$ is an initial condition.

$$I(a,0) = \int_0^\infty e^{-a^2u^2}\,du = \frac{1}{2}\sqrt{\pi}\,a^{-1}$$

$$I(a,b) = \frac{1}{2}\sqrt{\pi}\,a^{-1}e^{-2ab}$$

Finally, I have $\displaystyle\int_0^\infty e^{-a^2u^2-b^2u^{-2}}\,du = \frac{\sqrt{\pi}}{2a}e^{-2ab}$.

Briefly,

$$\int_0^\infty e^{-a^2u^2-b^2u^{-2}}\,du = a^{-1}\int_0^\infty e^{-v^2-a^2b^2v^{-2}}\,dv\text{, where } v = au$$

$$\frac{\partial[\int_0^\infty e^{-a^2u^2-b^2u^{-2}}\,du]}{\partial b} = -2\int_0^\infty e^{-v^2-a^2b^2v^{-2}}\,dv\text{, where } v = bu^{-1}$$

$$\frac{\dfrac{\partial[\int_0^\infty e^{-a^2u^2-b^2u^{-2}}\,du]}{\partial b}}{\int_0^\infty e^{-a^2u^2-b^2u^{-2}}\,du} = \frac{-2}{a^{-1}}\,.$$

So I have

$$\frac{\partial\left[\int_0^\infty e^{-a^2u^2-b^2u^{-2}}\,du\right]}{\partial b} = -2a\int_0^\infty e^{-a^2u^2-b^2u^{-2}}\,du$$

$$\frac{\partial e^{-ab}}{\partial b} = -2ae^{-ab}$$

so $\int_0^\infty e^{-a^2u^2-b^2u^{-2}}\,du = e^{-ab}$.

The initial condition has b=0 which means

$\int_0^\infty e^{-a^2u^2-b^2u^{-2}}\,du$ reduces to $\int_0^\infty e^{-a^2u^2}\,du$

and, of course, $\int_0^\infty e^{-a^2u^2}\,du = \frac{\sqrt{\pi}}{2a}$

so, in general, I have

$$\int_0^\infty e^{-a^2u^2-b^2u^{-2}}\,du = \frac{\sqrt{\pi}}{2a}e^{-ab}.$$

In order to demonstrate the usefulness of this result, consider the integral

$$\int_0^\infty t^{-1/2}e^{-q/t}e^{-\rho t}\,dt.$$

Changing the variable of integration necessitates the following substitutions: $t = u^2$, $t^{1/2} = u$, $dt = 2u\ du$

$$\int_0^\infty t^{-1/2}e^{-qu^{-2}}e^{-\rho u^2}\,2u\,du$$

$$2\int_0^\infty e^{-\rho u^2-qu^{-2}}\,du$$

$$2\int_0^\infty e^{-a^2u^2-b^2u^{-2}}\,du$$

where $a^2 = \rho$, $b^2 = q$, $a = \sqrt{\rho}$, and $b = \sqrt{q}$.

Using the result on page 15

$$2\int_0^\infty e^{-a^2u^2-b^2u^{-2}}\,du = 2[\frac{\sqrt{\pi}}{2a}e^{-2ab}]$$

and substituting for *a* and *b* yields $\frac{\sqrt{\pi}}{\sqrt{\rho}}e^{-2\sqrt{\rho q}}$ so I have

$$\mathrm{L}[t^{-1/2}e^{-q/t};\rho] = \frac{\sqrt{\pi}}{\sqrt{\rho}}e^{-2\sqrt{\rho q}}.$$

Consider another very important result. Take the derivative of the left hand side and the right hand side of the expression on page 15 with respect to *q*.

$$\mathrm{L}[t^{-1/2}(-\tfrac{1}{t})e^{-q/t};\rho] = \frac{\sqrt{\pi}}{\sqrt{\rho}}e^{-2\sqrt{\rho q}}(-2)(\frac{1}{2})(q\rho)^{-1/2}\rho$$

Multiply the LHS and the RHS by (-1) to yield

$$\mathrm{L}[t^{-3/2}e^{-q/t};\rho] = \frac{\sqrt{\pi}}{\sqrt{\rho}}e^{-2\sqrt{\rho q}}\frac{1}{\sqrt{q}\sqrt{\rho}}\rho$$

$$\mathrm{L}[t^{-3/2}e^{-q/t};\rho] = \frac{\sqrt{\pi}}{\sqrt{q}}e^{-2\sqrt{\rho q}}.$$

Appendix G

Does $\int_0^\infty g(t)dt$ equal 1?

$$\int_0^\infty g(t)\ dt = \int_0^\infty \frac{(W-c_0)}{\sigma\sqrt{2\pi t^3}} \exp[\frac{-(W-c_0-\mu t)^2}{2\sigma^2 t}]dt$$

$$\int_0^\infty g(t)\ dt = \frac{(W-c_0)}{\sigma\sqrt{2\pi}} \int_0^\infty t^{-3/2} e^{\frac{-[(W-c_0)^2+\mu^2t^2-2\mu t(W-c_0)]}{2\sigma^2 t}} dt$$

$$\int_0^\infty g(t)\ dt = \frac{(W-c_0)}{\sigma\sqrt{2\pi}} e^{\frac{2(W-c_0)\mu}{2\sigma^2}} \int_0^\infty t^{-3/2} e^{\frac{-1}{2}\frac{1}{\sigma^2 t}[(W-c_0)^2+\mu^2t^2]} dt$$

$$\int_0^\infty g(t)\ dt = \frac{(W-c_0)}{\sigma\sqrt{2\pi}} e^{\frac{2(W-c_0)\mu}{2\sigma^2}} \int_0^\infty t^{-3/2} e^{\frac{-1}{2}\frac{(W-c_0)^2}{\sigma^2}t^{-1}} e^{\frac{-1}{2}\frac{\mu^2 t}{\sigma^2}} dt$$

$$\int_0^\infty g(t)\ dt = \frac{(W-c_0)}{\sigma\sqrt{2\pi}} e^{\frac{(W-c_0)\mu}{\sigma^2}} \int_0^\infty t^{-3/2} e^{-qt^{-1}} e^{-\rho t} dt$$

where $q = \frac{(W-c_0)^2}{2\sigma^2}, \rho = \frac{\mu^2}{2\sigma^2}$.

According to Appendix F

$$\int_0^\infty t^{-3/2} e^{-qt^{-1}} e^{-\rho t} dt = \sqrt{\frac{\pi}{q}} e^{-2\sqrt{q\rho}}$$

$$\int_0^\infty g(t)\ dt = \frac{(W-c_0)}{\sigma\sqrt{2\pi}} e^{\frac{(W-c_0)\mu}{\sigma^2}} \sqrt{\frac{\pi}{\frac{(W-c_0)^2}{2\sigma^2}}} \cdot e^{-2[\frac{(W-c_0)^2}{2\sigma^2}\cdot\frac{\mu^2}{2\sigma^2}]^{1/2}}$$

$$\int_0^\infty g(t)\ dt = \frac{(W-c_0)}{\sigma\sqrt{2\pi}} e^{(W-c_0)\mu\frac{1}{\sigma^2}} \sqrt{\frac{2\sigma^2\pi}{(W-c_0)^2}} e^{-[\frac{1}{\sigma^2}](W-c_0)\mu}$$

$$\int_0^\infty g(t)\ dt = 1 \quad .$$

Appendix H

Consider the mean:

$$\int_0^\infty tg(t)dt = \int_0^\infty t\frac{(W-c_0)}{\sigma\sqrt{2\pi t^3}}\exp[\frac{-(W-c_0-\mu t)^2}{2\sigma^2 t}]dt$$

$$\int_0^\infty tg(t)dt = \frac{(W-c_0)}{\sigma\sqrt{2\pi}}\int_0^\infty t^{-1/2}e^{\frac{-[(W-c_0)^2+\mu^2t^2-2\mu t(W-c_0)]}{2\sigma^2 t}}dt$$

$$\int_0^\infty tg(t)dt = \frac{(W-c_0)}{\sigma\sqrt{2\pi}}e^{\frac{2(W-c_0)\mu}{2\sigma^2}}\int_0^\infty t^{-1/2}e^{-\frac{1}{2}\frac{(W-c_0)^2}{\sigma^2}t^{-1}}e^{-\frac{1}{2}\frac{\mu^2 t}{\sigma^2}}dt$$

$$\int_0^\infty tg(t)dt = \frac{(W-c_0)}{\sigma\sqrt{2\pi}}e^{\frac{(W-c_0)\mu}{\sigma^2}}\int_0^\infty t^{-1/2}e^{-qt^{-1}}e^{-\rho t}dt$$

where $q = \frac{(W-c_0)^2}{2\sigma^2}$, $\rho = \frac{\mu^2}{2\sigma^2}$.

Available in Appendix F is the mathematical fact that

$$\int_0^\infty t^{-1/2}e^{-qt^{-1}}e^{\rho t}dt = \sqrt{\frac{\pi}{\rho}}e^{-2(q\rho)^{1/2}} \text{ so that}$$

$$\int_0^\infty t\,g(t)dt = \frac{(W-c_0)}{\sigma\sqrt{2\pi}}e^{\frac{(W-c_0)\mu}{\sigma^2}}\frac{\sqrt{\pi}}{\sqrt{\mu^2}}\sqrt{2\sigma^2}e^{-2[\frac{(W-c_0)}{\sqrt{2}\sigma}]\frac{\mu}{\sqrt{2}\sigma}}$$

$$\int_0^\infty t\,g(t)dt = \frac{W-c_0}{\mu}\quad .$$

Appendix I

Given $y = \int_{g(x)}^{h(x)} f(x)\ dx$, according to Leibniz, the derivative of an integral where the limits are a function of the variable of differentiation is simply

$$\frac{\partial y}{\partial x} = \int_{g(x)}^{h(x)} f'\ x\ dx + \frac{\partial h(x)}{\partial x} \cdot f(h(x)) - \frac{\partial g(x)}{\partial x} \cdot f(g(x)) .$$

Appendix J

Consider the probability density function for trended Brownian motion

$$p(c_0, c; t_0, t) = \frac{1}{\sigma\sqrt{2\pi(t-t_0)}} e^{-(c-c_0-\mu(t-t_0))^2/2\sigma^2(t-t_0)}$$

by observation t_0 enters $p(c_0, c; t_0, t)$ as the mirror reflection of t and, consequently,

$$\frac{\partial p(\cdot)}{\partial t_0} = -\frac{\partial p(\cdot)}{\partial t}.$$

Appendix K

Commonly, trended Brownian motion is thought of as

$$dc_u = \mu\, du + \sigma\, dZ_u \quad .$$

However, integrating across the time horizon gives me

$$\int_0^t dc_u = \int_0^t \mu\, du + \sigma \int_0^t dZ_u$$

$$c_t - c_0 = \mu(t-0) + \sigma \int_0^t dZ_u$$

$$c_t = c_0 + \mu t + \sigma \int_0^t dZ_u$$

$$E(c_t) = c_0 + \mu t$$

$$V(c_t) = \sigma^2 V[\int_0^t dZ_u]$$

$$V(c_t) = \sigma^2 E[\int_0^t dZ_u]^2 \quad .$$

Ito's isometry allows me to write

$E[[\int_0^t dZ_u][\int_0^t dZ_u]]$ as $\int_0^t du$, so I have

$$V(c_t) = \sigma^2 \int_0^t du$$

$$V(c_t) = \sigma^2 t$$

$$c_t \sim N(c_0 + \mu t, \sigma^2 t).$$

With these characteristics, the pdf for the borrower's cash needs must be as follows

$$p(c_0, c; t) = \frac{1}{\sigma\sqrt{2\pi t}} e^{-(c-c_0-\mu t)^2/2\sigma^2 t} .$$

Appendix L

Let me demonstrate that $\mathrm{L}(f') = s\mathrm{L}(f)$.

Consider $\mathrm{L}(f') = \int_0^\infty e^{-st} f'(t)dt$

$$\int_0^\infty u dv = uv\Big|_0^\infty - \int_0^\infty v du$$
$$dv = f'(t)dt$$
$$u = e^{-st}$$
$$du = -se^{-st}dt$$
$$v = f(t).$$

The integral yields $\left[f(t)e^{-st}\Big|_0^\infty + \int_0^\infty f(t)se^{-st}dt \right]$

$$\mathrm{L}(f') = s\int_0^\infty f(t)e^{-st}dt - f(t=0).$$

Consequently, I have

$$\mathrm{L}(f') = s\mathrm{L}(f) - f(t=0)$$

or

$$\mathrm{L}(f') = s\mathrm{L}(f).$$

As an illustration consider $g'(t) = 3t^2$ and the Laplace transform of this derivative would be

$\mathrm{L}\ g'(t) = \int_0^\infty e^{-st} g'(t)dt$, I contend that $\mathrm{L}\ g'(t) = s\mathrm{L}\ g(t)$.

Given $\mathrm{L}\ g'(t) = \int_0^\infty e^{-st} 3t^2 dt$

$$uv\Big|_B^A - \int_B^A vdu = \int_B^A udv$$

$$u = e^{-st}$$

$dv = 3t^2 dt$, consequently I have

$$du = -se^{-st}dt$$
$$v = t^3$$

$$\mathrm{L}\ g'(t) = e^{-st}t^3\Big|_B^A + s\int_B^A t^3 e^{-st}dt$$

$$\mathrm{L}\ g'(t) = e^{-st}t^3\Big|_0^\infty + s\int_0^\infty t^3 e^{-st}dt$$

so I have $\mathrm{L}\ g'(t) = s\int_0^\infty g(t)e^{-st}dt$ or

$$\mathrm{L}\ g'(t) = s\mathrm{L}\ g(t)$$

which is what I wanted to show.

Appendix M

A Laplace transform of a linear combination of the functions $f(t)$ and $g(t)$ is the linear combination of the Laplace transform of $f(t)$ and the Laplace transform of $g(t)$.

Consider $\mathrm{L}\ af(t)+bg(t) = a\mathrm{L}\ f(t) + b\mathrm{L}\ g(t)$

because

$$\mathrm{L}\ af(t)+bg(t) = \int_0^\infty e^{-st}\ af(t)+bg(t)\ dt$$

$$\mathrm{L}\ af(t)+bg(t) = a\int_0^\infty e^{-st} f(t)dt + b\int_0^\infty e^{-st} g(t)dt$$

$$\mathrm{L}\ af(t)+bg(t) = a\mathrm{L}\ f(t) + b\mathrm{L}\ g(t)\ .$$

Appendix N

Consider a detailed determination of $\text{Prob}\{T_R < T_M < T_{R+G}\}$ where T_M is the time of maturity of the credit line which is non-stochastic. Both T_R and T_{R+G} are random variables. T_R is the time to when the cash needs of the borrower exceeds the bank's first source of loan funding R, bank reserves. T_{R+G} is the time to when the loan drawdown exceeds $R+G$, all the bank's assets on hand to fund the loan commitment.

I know

$$\text{Prob }(C_{T_M} > R) = \int_R^{\infty} f(c_{T_M})dc_{T_M} = \int_R^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 T_M}} e^{-(c-\mu T_M)^2/2\sigma^2 T_M} dc$$

$$\text{Prob }(T_R < T_M) = \int_0^{T_M} g_R(t)\,dt = \int_0^{T_M} \frac{R-0}{\sqrt{2\sigma^2\pi t^3}} e^{-(R-\mu t)^2/2\sigma^2 t} dt$$

$$\text{Prob }(C_{T_M} > (R+G)) = \int_{R+G}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 T_M}} e^{-(c-\mu T_M)^2/2\sigma^2 T_M} dc$$

$$\text{Prob }(T_{R+G} < T_M) = \int_0^{T_M} \frac{(R+G)-0}{\sqrt{2\sigma^2\pi t^3}} e^{-(R+G-\mu t)^2/2\sigma^2 t} dt \ .$$

I also know

$$\text{Prob }(C_{T_M} > R) = \text{Prob }(T_R < T_M)$$

$$\text{Prob }(C_{T_M} > (R+G)) = \text{Prob }(T_{R+G} < T_M)$$

$$\text{Prob }(C_{T_M} < (R+G)) = \text{Prob }(T_M < T_{R+G}) \ .$$

From Bayes, I have

$$\text{Prob }(T_R < T_M < T_{R+G}) = \text{Prob}\{T_M < T_{R+G} \,|\, T_M > T_R\} \cdot P(T_M > T_R)$$

$$\text{Prob }\{T_M < T_{R+G} \,|\, T_M > T_R\} = \text{Prob}\{C_{T_M} < (R+G) \,|\, C_{T_M} > R\}$$

$$\text{Prob}\{T_M > T_R\} = \text{Prob}\{C_{T_M} > R\} = \int_R^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 T_M}} e^{-(c-\mu T_M)^2/2\sigma^2 T_M} dc \ .$$

So I now have

$$\text{Prob }(T_R < T_M < T_{R+G}) = \text{Prob}\{C_{T_M} < (R+G) \,|\, C_{T_M} > R\} \cdot \text{Prob}\{C_{T_M} > R\}$$

$$\text{Prob }(T_R < T_M < T_{R+G}) = \left[\frac{\text{Prob}\{C_{T_M} < (R+G)\} - \text{Prob}\{C_{T_M} < R\}}{1 - \text{Prob}\{C_{T_M} < R\}} \right] \cdot \text{Prob}\{C_{T_M} > R\}$$

$$\text{Prob }(T_R < T_M < T_{R+G}) = \left[\frac{\text{Prob}\{C_{T_M} < (R+G)\} - \text{Prob}\{C_{T_M} < R\}}{\text{Prob}\{C_{T_M} > R\}} \right] \cdot \text{Prob}\{C_{T_M} > R\}$$

$$\text{Prob }(T_R < T_M < T_{R+G}) = \text{Prob}\{C_{T_M} < (R+G)\} - \text{Prob}\{C_{T_M} < R\}.$$

Consequently,

$$\text{Prob }[T_R < T_M < T_{R+G}] = \int_{-\infty}^{R+G} f(c_{T_M})\, dc_{T_M} - \int_{-\infty}^{R} f(c_{T_M})\, dc_{T_M}$$

or

$$\text{Prob }[T_R < T_M < T_{R+G}] = P\{R < C_{T_M} < (R+G)\} \ .$$

Appendix O

$$\text{Prob }(T_R < T_M < T_{R+G}) = \text{Prob}\{T_M < T_{R+G} \,|\, T_M > T_R\} \cdot \text{Prob}(T_M > T_R)$$

$$\text{Prob }(T_R < T_M < T_{R+G}) = \left[\frac{\text{Prob}[T_M < T_{R+G}] - \text{Prob}[T_M < T_R]}{\text{Prob}(T_M > T_R)} \right] \cdot \text{Prob}(T_M > T_R)$$

$$\text{Prob }(T_R < T_M < T_{R+G}) = \text{Prob}[T_M < T_{R+G}] - \text{Prob}(T_M < T_R)$$

$$\text{Prob }(T_R < T_M < T_{R+G}) = \int_{T_M}^{\infty} \frac{(R+G)}{\sqrt{2\pi\sigma^2 t^3}} e^{-(R+G-\mu t)^2/2\sigma^2 t} dt - \int_{T_M}^{\infty} \frac{R}{\sqrt{2\pi\sigma^2 t^3}} e^{-(R-\mu t)^2/2\sigma^2 t} dt.$$

$$\text{Prob }(T_R < T_M < T_{R+G}) = \left[1 - \int_0^{T_M} g_{R+G}(t) dt \right] - \left[1 - \int_0^{T_M} g_R(t) dt \right]$$

$$\text{Prob }(T_R < T_M < T_{R+G}) = \int_0^{T_M} g_R(t) dt - \int_0^{T_M} g_{R+G}(t) dt$$

$$\text{Prob }(T_R < T_M < T_{R+G}) = \text{Prob}\{C_{T_M} > R\} - \text{Prob}\{C_{T_M} > (R+G)\}$$

$$= \left[1 - \int_{-\infty}^{R} f(c_{T_M}) dc_{T_M} \right] - \left[1 - \int_{-\infty}^{R+G} f(c_{T_M}) dc_{T_M} \right]$$

$$= \int_{-\infty}^{R+G} f(c_{T_M}) dc_{T_M} - \int_{-\infty}^{R} f(c_{T_M}) dc_{T_M}$$

Alternatively

$$\text{Prob }(T_R < T_M < T_{R+G}) = \text{Prob}\{C_{T_M} < (R+G) \big| C_{T_M} > R\} \cdot \text{Prob}\{C_{T_M} > R\}$$

$$\text{Prob }(T_R < T_M < T_{R+G}) = \left[\frac{\text{Prob}\{C_{T_M} < (R+G)\} - \text{Prob}\{C_{T_M} < R\}}{1 - \text{Prob}\{C_{T_M} < R\}} \right] \cdot \text{Prob}\{C_{T_M} > R\}$$

$$\text{Prob }(T_R < T_M < T_{R+G}) = \left[\frac{\text{Prob}\{C_{T_M} < (R+G)\} - \text{Prob}\{C_{T_M} < R\}}{\text{Prob}\{C_{T_M} > R\}} \right] \cdot \text{Prob}\{C_{T_M} > R\}$$

$$\text{Prob }(T_R < T_M < T_{R+G}) = \text{Prob}\{C_{T_M} < (R+G)\} - \text{Prob}\{C_{T_M} < R\}.$$

Consequently,

$$\text{Prob }[T_R < T_M < T_{R+G}] = \int_{-\infty}^{R+G} f\,(c_{T_M})\; dc_{T_M} - \int_{-\infty}^{R} f\,(c_{T_M})\; dc_{T_M}$$

or

$$\text{Prob }[T_R < T_M < T_{R+G}] = P\{R < C_{T_M} < (R+G)\} \quad .$$

Appendix P

Changes in capital across the simulations in Table 2 necessarily change the level of bank operations and, consequently, impact R^* and G^* as the bank seeks to maximize its intertemporal profits. But just how different are these optimal solutions for bank liquidity from the recalibrations of R^* and G^* if regulators insisted that the recorded changes in bank scale must preserve the existing likelihood of a liquidity shortage. That is changes in intermediary capital can support increases in bank operations or occasion a contraction in the LHS of the bank's tee account but the risk of a depletion of liquid assets must be unchanged. Characterizing the behavior of bank liquidity with this objective in mind is simply a matter of setting the differential change in RLS to zero and then invoking the implicit function theorem as in

$$dRLS = 0 = \frac{\partial RLS}{\partial q} dq + \frac{\partial RLS}{\partial k} dk$$

so that

$$\frac{dq}{dk} = (-1)\frac{\dfrac{\partial RLS}{\partial k}}{\dfrac{\partial RLS}{\partial q}} = (-1)\frac{\int_0^{T_M} \mathrm{g_q}(t)\left[\dfrac{\mu t\text{-}q}{\sigma^2}\right]\dfrac{1}{\theta}dt}{\int_0^{T_M} \mathrm{g_q}(t)\left[\dfrac{1}{q} + \dfrac{\mu t}{\sigma^2 t} - \dfrac{q}{\sigma^2 t}\right]dt}$$

for $q = (R + G)$ since a bank's reserves plus its government securities best fits the Federal Reserve's description of "unencumbered, high quality assets". Combining the solution to the expression above with the differential change in bank capital (recorded in Table 2) yields

$$dq^{**} = \frac{\partial q}{\partial k} dk$$

the optimal change in the intermediary's stored liquidity q^{**} for $dRLS = 0$.

From Table 5, it is clear that the changes in "RLS" preserving stored liquidity follow changes in k^*. (The direction of changes in the profit maximizing reserves and government securities have been detailed earlier in this paper.) Optimal bank capital increases in three of the six comparative static simulations and $d(R+G)^{**}$ is positive in those experiments. When k^* falls the constant "RLS" level of liquidity decreases. Intuitively increases in intermediary equity occasion an increase in W^* and bank liquidity is augmented to support the takedown of the enhanced credit line so that $dRLS = 0$. When capital contracts, W^* retreats and $d(R+G)^{**}$ falls since less liquidity is needed to fund the uncertain usage of W^* and preserve the "RLS". There are differences in $d(R^*+G^*)$ and $d(R+G)^{**}$ and while the differences are not large relative to the magnitude of (R^*+G^*) the changes are in opposite directions in 5 of the 6 simulations. Whether or not the differences are significant could best be determined by the implication the each change has for $E_{(R^*+G^*)}[T]$ and $E_{(R+G)^{**}}[T]$, respectively. The expected time to depletion for $(R+G)^{**}$ is stable throughout the comparative static simulations never straying too far from the base case result of 149.58 days. Given that $d(R+G)^{**}$ is an artifact of setting $dRLS$ equal to zero, the stability of the expected time to exhausting stored liquidity is hardly a surprise. In fact the only significant deviations of the expected time to depletion of bank reserves, primary and secondary, should be a pleasant surprise to bank regulators. In Table 6, parametric changes for two of the simulations dramatically increase the profit maximizing level of R^*+G^* and occasion increases in $E_{(R^*+G^*)}[T]$ to 180.45 days and to 163.83 days.

Table P1: RLS Preserving Changes in Bank Liquidity Associated with Optimal Changes in Intermediary Capital (B=billions)

Parameter	Given a 10% Increase in r_1	Given a 10% Increase in r_2	Given a 10% Increase in vc_G	Given a 10% Increase in D	Given a 10% Increase in ρ	Given a 10% Decrease in γ
dk^*	\$0.011B	\$-0.205B	\$0.028B	\$0.390B	\$-0.212B	\$-0.016B
$d(R^* + G^*)$	\$-0.852B	\$4.035B	\$-0.764B	\$2.260B	\$0.541B	\$0.029B
$d(R+G)^{**}$	\$0.084B	\$-0.573B	\$0.215B	\$1.092B	\$-0.627B	\$-0.123B

Table P2: Expected Time to Depletion for RLS Preserving Changes in Bank Liquidity (B=billions, d=days)

Parameter	Given a 10% Increase in r_1	Given a 10% Increase in r_2	Given a 10% Increase in vc_G	Given a 10% Increase in D	Given a 10% Increase in ρ	Given a 10% Decrease in γ
dk^*	\$0.011B	\$-0.205B	\$0.028B	\$0.390B	\$-0.212B	\$-0.016B
$E_{(R^*+G^*)}[T]$	145.24d	180.45d	145.06d	163.83d	151.47d	147.87d
$E_{(R+G)^{**}}[T]$	150.19d	148.27d	151.13d	152.79d	147.89d	148.71d

End Notes

[1] Bank capital has attracted the interest of academic researchers for a long time. As Berger, Herring, and Szego (1995) suggest, some of the most fundamental bank capital questions revolve around such intertwined issues as the optimal amount of bank capital , the amount of capital that regulators require banks to keep, and the amount of capital that banks actually choose to keep. Many researchers have analyzed how bank capital requirements affect the financial system. Kim and Santomero (1988) suggest that the effect of capital requirements on bank safety depends on the intermediary's attitude toward risk. That is, some banks with relatively little risk aversion may respond to greater capital requirements (that may decrease return on equity) by increasing the riskiness of their asset portfolio while other more risk averse banks may not be impacted at all. Jeitschko and Jeung (2005) challenge the idea that well-capitalized banks are less inclined to increase asset risk and assume three agents --- deposit insurers, shareholders, and managers--- collectively determine bank riskiness. (Such banks with strong capital will are not inclined to increase asset risk because the option value of deposit insurance decreases with capitalization.) Kopecky and VanHoose (2004) analyze the potential conflict between bank regulators and monetary authorities. In particular, they make capital an endogenous variable and examine loan expansion assuming both binding and nonbinding capital requirements. VanHoose (2007) summarizes theories of bank behavior under capital regulation and suggests that the theoretical foundations for capital requirements are very questionable. That is, greater capital requirements do not necessarily lead to safer banks and he argues that more research is needed. (Given the complaints that many bankers make about attempts by regulators to require greater capital, it is surprising that Flannery and Rangan (2008) find banks held 75% above regulatory capital minimums in the early part of the 21st century. Furthermore, Allen, Carletti, and Marquez (2009) also find that banks hold more capital than regulators require which is attributed to monitoring theory where asset side discipline, as opposed to threats of deposit runs, can be very important.)

[2] From October of 2008 to July of 2009, bank loans fell at an average weekly rate of .05%. While it is unclear how much of this decline is due to the supply side of the market and how much is attributable to a leftward shift in the demand schedule, there are reasons to believe that loan demand did contract. Besides the weak economy, the demand for bank loans may have fallen in this period because of the success of programs initiated by the Federal Reserve to support market-based financing. For example, the Federal Reserve announced the Commercial Paper Funding Facility (CPFF) on October 7, 2008, which became fully operational on October 27, 2008. It served as a lender of last resort for the commercial paper market, with the Federal Reserve directly financing purchases of commercial paper. This program gave non-financials easier access to commercial paper funding, and they had less of a need to rely on credit lines from banks. We interpret this market intervention by the Fed as providing borrowers with another source of funds. Consequently, we record the development of the "CPFF" as a change in the elasticity of the demand for loans and characterize the impact of the change upon bank capital and depository assets.

[3] We model the cash needs of the borrower as trended Brownian motion,

$$p(c_o, c; t) = \frac{1}{\sigma\sqrt{2\pi t}} e^{-(c-c_o-\mu t)^2/2\sigma^2 t},$$

so that the loan applicant's cash demands at any point in time "t", are normally distributed. (The borrower's cash needs have unbounded support in this paper. Consequently, problems of negative cash needs can arise. Throughout the paper, the non-negative constraint is ignored for simplicity. The probability of a borrower having negative cash needs can be made arbitrarily small by the appropriate choice of the underlying statistical parameters that we use to characterize $p(c_o, c; t).$) If borrower demands are expected to grow rapidly then the parameter μ will be relatively large. If there is a great deal of uncertainty regarding the loan recipient's financial needs then σ will be large. Using $p(c_o,c;t)$, we are able to characterize the borrowing from the bank over time. In addition, the statistical properties of trended Brownian motion allow us to derive a probability density function for the time to depletion of the loan account, as well as likelihoods for the time to exhausting the respective bank assets that fund the loan takedown.

[4] When a bank agrees to fund a loan commitment, it faces many risks as it seeks to maximize the expected profit associated with the issuance of a line of credit. Since loan takedown by the customer is stochastic, a bank must stand ready to provide to the borrower an uncertain amount of cash, up to the contracted maximum amount. On the liability side of the balance sheet, the level of customer deposits varies over time and adds to the uncertain availability of liquid assets. In addition, the vagaries of the marketplace impact the cost of converting illiquid assets into cash for the loan recipient. Finally, there may be times when external capital needed to provide funds for a loan takedown may be prohibitively expensive or not available at all. The earliest work to specifically address these problems is by Deshmukh, Greenbaum and Kanatas (1982). They model a bank as a seller of loan commitments in alternate funding environments to solve the "simultaneous lending-financing problem". The bank funds loans using a mix of deposits and purchased money. The authors argue that as the bank maximizes profits, it must consider the cost of funding along with the returns on assets and loans to find its optimal commitment volume. Kashyap, Rajan, Stein (2002) provide evidence that banks are the most efficient providers of liquidity to commercial firms. In a new approach to studying the bank's funding risk, the authors argue that aggressive credit line usage by borrowers during economic downturns is accompanied by high deposit growth as non- commercial customers seek a safe haven for their cash. During economic booms, commercial firms generate large amounts of cash internally and do not access their credit lines. At the same time, depositors are more likely to hold alternative financial assets such as money market accounts, which would lead to a decrease in the level of bank deposits. Therefore, when banks are in the greatest need of liquidity, customers happen to provide the necessary deposits, and when surplus funding would otherwise exist, deposit disintermediation occurs. Gatev and Strahan (2006) discuss and extend the KRS model by using commercial paper (CP) spreads (over the T-Bill rate) to analyze loan commitment exposure by banks. When this spread widens, it becomes more expensive for firms to procure short term funding in the CP market, thereby restricting corporate liquidity. In this environment, existing credit lines become the preferred, if not the only, source of borrowing. The authors find that banks increase liquid asset holdings during periods of higher risk (larger CP spreads) in order to fund anticipated takedowns. During times of elevated risk, banks can gather funds at lower rates than competitors since their customers are anxious to park their financial assets in the safest location possible. Gatev, et al (2007) examine bank liquidity risk by testing if the equity market distinguishes between banks that have low deposit funding and high loan commitment exposure, and those that do not. By using bank equity volatility as a proxy for market uncertainty, the authors' empirical findings confirm that investors can differentiate between banks that have heterogeneous levels of funding risk. Landskroner and Paroush (2008) present a model of asset-liability management by a bank that offers loan commitments. If a bank must be ready at all times during the commitment contract to provide cash to its borrowers, there can arise a potential liquidity gap that banks must be willing to breach with either the conversion of assets or securing external funding. The authors show that a bank's liquidity risk increases as competition in the banking sector grows. This change in bank competition results in greater demand elasticity on the part of the borrowers and the supply elasticity of depositors, which leads to more balance sheet volatility. The credit line literature also includes papers that examine liquidity risk from the borrower's point of view. Sufi (2008), in his study of corporations with used and unused lines of credit, demonstrates that firm cash flow is a strong predictor of whether a firm uses credit lines or cash in managing liquidity. The authors find that firms with higher credit risk need increased cash flow in order to establish a credit line; they also show that loan takedown is positively correlated with whether a firm faces capital market frictions such as illiquid debt markets, burdensome regulations, or high transaction costs. These frictions make it more costly to issue debt in public markets and encourage borrowing from the credit line. In a study of international corporate credit markets, Lins, Servaes and Tufano (2008) analyze what type of firm is more likely to use lines of credit to maintain corporate liquidity. Firms can choose between holding high levels of cash, keeping a line of credit open or a combination of both. According to this study, firms (on average) maintain a line of credit equal to 15% of book assets.

[5] Please see Appendix B for a heuristic derivation of the Kolmogorov diffusion equation.

[6] Please see Appendix C and D for demonstrations that $P(c_0, W; t)$ and $g(t; c_0, W)$, respectively, satisfy Kolmogorov's diffusion equation.

[7] Please see Appendix B for a demonstration of the fact that trended Brownian motion satisfies the Kolmogorov diffusion equation.

[8] Please see Appendix E for a brief tutorial on Laplace transforms.

[9] That is $A_2 = e^{-W}$.

[10] Clearly when $c_0 = W$, then $\lambda(W) = 1$. In general, the inverse of $\lambda(W)$ is

$$g(t; c_0, W) = \frac{W - c_0}{\sigma\sqrt{2\pi t^3}} \exp[\frac{-(W - c_0 - \mu t)^2}{2\sigma^2 t}].$$

If W is the starting value for c_0 then $\lambda(W)$ is equal to 1, so that we have $g(t; c_0, W)$=0.

[11] Even the most casual reader will recognize that what we call "time to loan depletion" is what statisticians call the "time to first passage". Please see Bailey (1964), Barlett (1955), Bharucha-Reid (1960), Darling and Siegert (1953), Doob (1953), Feller (1957), Parzen (1962) and Uhlenbeck and Ornstein (1930) for the background analysis that enabled us to derive the probability density for the time to loan depletion.

[12] Please see appendix I and then appendix J for confirmation that $\int_0^\infty g(t)\,dt = 1$ and that $\int_0^\infty t\, g(t)\,dt = \frac{W - c_0}{\mu}$.

[13] While γ is included in our intertemporal profit function, it is not clear that the non-usage fee is a decision variable in this particular analysis. Shockley and Thakor (1997) maintain, along with Greenbaum, et al (1991), that γ serves as a device to sort firms into different borrower "types", which gives the bank information about the firm that cannot otherwise be revealed. Other research, such as Denis, et al (2000), finds that γ is used in loan contracts by firms that have multiple commitments, since they will likely have more unused loan capacity. The authors also argue that firms that choose longer maturities and have greater overall leverage tend to enter into contracts with a non-usage fee. Boot, et al (1987) illustrate how γ serves to mitigate underinvestment by the borrower. Thakor and Udell (1987) explain how non-usage fees help banks overcome information asymmetry problems. Clearly these papers assign an informational role to γ, not one that is related to credit line pricing per se. Consequently, in our analysis, the value of γ is given exogenously.

[14] The recent credit crisis has highlighted the importance of overnight lending by financial institutions. A Bloomberg article ("Money Rates Double Amid Global Credit Seizure", 9/16/2008) reports that overnight lending rates have risen dramatically in the face of the recent disruptions in the financial markets, from 3.33 % to 6.44 %, its biggest jump in seven years. The Financial Times ("Overnight Lending Still in Lockdown", 10/4/2008) confirms that the recent upheaval in the credit markets has impacted the interbank lending market, leading to a great deal of instability in the banking sector. Overnight lending typically occurs in the "Fed funds" market and allows lenders to earn a rate of return on otherwise idle funds. Fed funds are overnight borrowings by banks to maintain appropriate levels of reserves at the Fed. Funds available for investment in overnight lending are represented by R in our analysis. The bank in our model earns a modest rate of return on this highly liquid asset. In practice, many different financial institutions participate in the Fed funds market, including commercial banks, thrift institutions, agencies and branches of foreign banks in the United States, federal agencies, and government securities dealers. Most of these transactions occur on Fedwire, a transfer system operated by the Federal Reserve. From 2000-2007, over $2 trillion per day passed through Fedwire, on average. To put this in perspective, the total average daily volume of *global* foreign exchange trading is $3.2 trillion. In 2008, however, the daily volume of transfers on Fedwire has surged to $3 trillion per day on average (National Settlement Services Quarterly Data, Q2 report). This growth in daily transfers reflects the recent distress in the credit markets, leading to a propensity for banks to lend and borrow in highly liquid, low-risk assets at overnight rates. This increase in trading volume is caused by rightward shifts in both the demand *and* supply schedules for short-term funds, with the former shift being larger than the latter. On the demand side, banks that previously found accessible sources of liquidity for the left hand side of their balance sheets are forced to borrow Fed funds just to satisfy reserve requirements. On the supply side, banks are hesitant to lend money in long-term markets, due to risk aversion in the banking sector after a series of high-profile bank failures

(such as WaMu) in the summer of 2008, and thus they move their assets into overnight markets. The academic literature has analyzed the overnight market for funds. Furfine (2001) studies the role of credit risk in overnight loan pricing. Bartolini, et al (2005) describes the microstructure of this market. Ashcraft and Duffie (2007) detail the pricing and allocation of overnight loans with an analysis of the determinants of the likelihood of institutional lending at a particular time and day.

[15] Clearly the bank's expected profits depend upon the time to when R, G and the credit line are each depleted. Using the statistical properties of the borrower's cash needs, we find that the time to depletion to each of these three accounts satisfy Kolmogorov's diffusion equation. The solution to each of the three differential equations allows us to determine the probability density function of the time to depletion to each account; the loan account, R and G. Since solving for the time to exhaustion of any one of the accounts is much like solving for either of the other two, our analysis here will emphasize the time to depletion of the bank's credit lines and the time to depletion of the remaining accounts will just be utilized in this paper without explanation.

[16] Over the last 30 years, due to advances in banking technology, banking competition, and the globalization of financial markets, bank asset conversion costs have certainly fallen. However, in the commercial banking industry asset liquidation costs still exist and they continue to be an aspect of academic models used in banking research. Stein (1998) presents a model in which a bank has two assets, loans and securities. Loans are costly to convert to cash while securities have no liquidation costs. The bank is subject to random customer demands for cash. If the bank only lends, then it may eventually be forced to sell some portion of the loan and pay liquidation costs to accommodate customer liquidity needs. If the bank holds only government securities, for example, it can easily convert the securities to cash, but the intermediary receives a lower rate of return on the securities. The bank's dilemma is how it will allocate its portfolio of loans and securities over time as it manages this trade-off between higher rates of return and potential conversion costs. Diamond and Rajan (2005) analyze how the asset side of a bank's balance sheet can be a source of liquidity shocks. The authors assume that loans, which represent the bank's investment "projects", cannot be sold for full value before maturity. Since the bank must respond to demands for immediate liquidity by depositors, the bank may have to sell a loan at a discount in order to fund these cash needs. The reduced value of the loan, of course, represents a liquidation cost to the bank. Wagner (2007) develops a model in which a bank holds two assets, loans and reserves. The author derives the optimal configuration of bank assets when the depository institution seeks to maximize expected profit and yet manage the likelihood of a bank run. In Wagner's model, proportional conversion costs impact the first order conditions for the bank's decision variables.

[17] In order to convert treasuries into funds for financing a loan takedown, a bank must stand ready to make transactions in the capital markets on a daily basis. The decisions involved in effecting these trades are likely made by the bank's own financial managers located on a trading desk at corporate headquarters. Whether a bank actually makes daily trades for billions of dollars, or a monthly trade for as little as a million dollars, the bank must maintain an operation capable of gathering market information and executing trades to fulfill the needs of loan commitment customers. Also, it is likely a support staff is employed by the bank which assists the asset management team. This "back office" team of financial professionals is another fixed cost to any bank that converts assets into cash for a loan takedown. The banking literature includes several papers that discuss the fixed costs that financial institutions must pay in order to participate in the business of lending money. Pulley and Humphrey (1993) study cost complementarities within a bank's lending operation, which include branch offices, computer technology, and accounting services as necessary fixed costs. The authors also study the economies of scope in different banking markets and how the aforementioned fixed costs are distributed. Song and Thakor (2007) construct a fragility model to demonstrate how banks optimally match their assets and liabilities in the face of withdrawal risk by customers. In their model, the bank must pay a fixed investment (cost) to provide "liquidity services", or core deposits. The authors argue that the fixed costs represent "the bank's investment in the infrastructure of branches" and "transaction services".

[18] FC_2 is set equal to zero in the simulations due to the difficulty of providing a numeric characterization of the fixed costs of converting G into R.

[19] Please see Appendix N for a detailed explanation.

[20] Media coverage of the credit crisis of 2008 has highlighted the importance of the interbank lending market. The New York Times ("No Sign That Lenders Will Thaw" 10/12/2008) reports that the stability of global credit markets depend greatly on interbank lending rates. The BBC News ("Interbank Lending in Deep Freeze" 10/10/2008) explains how short-term lending markets have seized up in the face of multiple bank failures. Banks borrow and lend money in the interbank market in order to manage liquidity and meet reserve requirements. In particular, banks need to hold an adequate amount of liquid assets to accommodate both loan takedowns by borrowers and potential withdrawals by depositors; otherwise, it will need to borrow money in the interbank market to cover the shortfall. Banks with excess liquidity will lend money in the interbank market, receiving interest on the assets. Interbank loans can be short-term, such as "overnight" lending, or longer term, up to one year. There is a wide range of published interbank rates, including the LIBOR (the London Interbank Borrowing Rate), which is based on the average rates on loans made within the London interbank market. In our model, the two liquid assets that can be easily converted to fund borrower takedown are R, which represents excess reserves held by the bank available for overnight lending and G, short-term Treasury securities. The bank can choose to invest excess reserves at r_1, which would earn the lowest rate but offers the lowest conversion cost. In order to receive a higher rate of return, the bank can invest in Treasury securities and make r_2; generally the bank holds both R and G. If the loan takedown by the borrower exceeds the bank's stored liquidity, then the bank turns to interbank markets to raise funds. Due to this shortfall, the bank secures a term loan in the amount of B at r_3 (typically one, three, six, or twelve months in maturity). Shorter term borrowing would not be appropriate in this case, since the bank's intertemporal needs may be as long as the existing time to maturity of the loan commitment. Many academic researchers have discussed the importance of interbank lending markets. Rochet, et al (1996) present a theoretical model of interbank lending. King (2008) discusses credit risk and interbank lending. Carletti, et al (2007) explains how mergers influence interbank markets and aggregate liquidity. Dinger and Hagen (2007) show how interbank lending lowers risk for borrowing banks. Michaud and Upper (2008) use LIBOR rates to explicate the relationship between credit risk and liquidity in interbank rates.

[21] Many of our choices for the illustrative values used in the simulations can be challenged. However, it is our intent to demonstrate that the model obtains numeric solutions to the FOCs and to establish the robustness of those solutions.

[22] Newton's approach to the numeric solution of (24) was chosen for its simplicity and its convergence properties.