Hybrid Functional Autoregressive Modelling of Non-parametric Density for Improved Value-at-Risk Analysis

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Version: Feb. 2012

Abstract

FAR modelling of the non-parametric density function is proposed to improve Value-at-Risk (VaR) analysis by taking into account the relative advantages of parametric and non-parametric models in a hybrid manner. In particular, this approach enables us to use the intraday information for forecasting the time-varying daily return density function. It is well-established that VaR forecasts obtained from parametric and non-parametric models involve a trade-off between minimising the associated economic costs and providing the valid coverage of VaR. The Monte Carlo simulation study and empirical evaluations of VaR, based on thirty components of the Dow Jones Industrial Average and their equal weighted portfolio, clearly demonstrate that the overall performance of the proposed hybrid model is superior to those of both the parametric and the non-parametric models in terms of several (sometimes conflicting) criteria. In particular, the proposed hybrid approach is shown to simultaneously increase coverage ability, reduce economic costs and enhance the statistical validity. Hence, it is recommended to use the hybrid functional autoregressive non-parametric density approach (along with another hybrid model called the filtered historical simulation approach) for improving an internal VaR model for both regulators and banks in a fair and satisfactory manner.

JEL Codes: G11; C51; C52; C53
Keywords: Financial Risk Management, Value-at-Risk, Hybrid Approach, Functional Autoregressive Model.

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1 Introduction

Value-at-Risk (VaR) has been used as the central measure of risk in the banking regulatory framework. VaR was popularised by the so-called RiskMetric of J. P. Morgan. It was first established in 1989, when Dennis Weatherstone, the chairman of J. P. Morgan, asked for a daily report measuring and explaining the risk of his firm and nearly four years later in 1992, J. P. Morgan launched RiskMetrics. Moreover, VaR was more popular as a risk measure chosen by investment banks to measure their portfolio risk for the benefit of banking regulators (RiskMetrics, 1996). VaR was also strongly recommended by regulatory organisations. In 1997, the U.S. Securities and Exchange commission ruled that public corporations must disclose quantitative information about their derivatives activity. Major banks decided to comply with this rule by including VaR information in their financial statements. Furthermore, in 1999, the worldwide adoption of the Basel II Accord stimulated financial institutions to use VaR. Nowadays, most major banks report the VaR forecast to the regulator who supervises the bank's risk management.

The urgent need of an accurate internal VaR model was demonstrated during the 2007–2009 global financial crisis. The role of financial risk management has become a notable concern for survival at both the micro- (firm) and macro-level (nationwide). Furthermore, it is critical for banks to accurately measure and disclose the level of risk in their business that is of interest to their investors, creditors and regulators. While the use of VaR as a summary financial risk measure is appealing in theory and practice, difficulties arise in searching for the best VaR model which minimises economic cost and maximises coverage ability.

There have been major developments of the VaR model in the last two decades (see Jorion, 2006 for an excellent survey). However, challenges remain due to the difficulty of modelling a time-varying non-normal return distribution as well as forecasting an extreme value. The previous literature on the VaR modelling falls into three groups: parametric, non-parametric and hybrid approaches. First, there is the parametric approach such as GARCH models and extreme value distribution models (hereafter, EVT models). However, it is well known that the parametric approach is intrinsically bound to suffer from the following limitations. Particularly, there are uncertainties in (i) the distribution assumption, (ii) the time dependent specification of the higher order moments, and (iii) the computational complexities of the numerical optimisation algorithm. Extensive empirical evidence shows that stock return distributions are fat-tailed.[1] Hence, a normal distribution always underestimates the risk of the fat-tailed distributions observed in financial time series. For example, it is well-established that Gaussian GARCH underestimates VaR (Netftci, 2000). RiskMetrics (hereafter RM) is a typical example using the normal distribution (Johansson et al., 1999). The Student's t-distribution is thus a preferable choice. It is however a symmetric distribution so that it cannot capture the asymmetry of the asset return distribution (Giot and Laurent, 2003; Mittnik and Paolella, 2000). Recent studies model the fat-tail and the asymmetry of the asset return distribution by allowing for higher-order moments with a specific parametric distribution such as the skewed generalised error distribution (Theodossiou, 1998). Furthermore, some progress has been made in modelling the dynamics of the higher order moments

[1] Mandelbrot (1963) first captured the fat-tail of stock returns distribution and criticised the extensive reliance on the normality assumption for asset pricing and investment theory. Fama (1963, 1965), Clark (1973) and Blattberg and Gonedes (1974) proposed allowing the stock returns to be non-normal and developing the statistical modelling of stock returns as draws from a fat-tailed distribution.

within the GARCH framework; GARCH with skewness (GARCHS, Harvey and Siddique, 1999), GARCH with kurtosis (GARCHK, Brooks et al., 2005) and GARCH with both skewness and kurtosis (GARCHSK, Jondeau and Rockinger, 2003). Bali et al. (2008) demonstrate that the accuracy of a VaR forecast improves significantly when conditional higher order moments are modelled by adopting an asymmetric distribution function. However, the contemporaneous and the time-dependent structure among the moments are too complicated to be specified completely (Jondeau and Rockinger, 2003). EVT models consider the extreme value distribution given a block size.[2] The Generalised Extreme Value (GEV) distribution and the Generalised Pareto Distribution (GPD) are popular and frequently employed in VaR modelling. Since EVT models provide a conservative VaR, they are likely to overestimate VaR, leading to an expensive cost.

Second, a non-parametric model is popular among practitioners. A simple historical simulation (hereafter, HS) is found to be the industry standard (Perignon et al., 2008; Perignon and Smith, 2009). In spite of its popularity, severe problems are reported in the literature Barone-Adesi et al. (1999, 2002); Jorion (2006) due to its ignoring the time-varying nature of the asset return distribution. Since HS is intrinsically sensitive to the existing state, it cannot provide a valid VaR when the underlying regime changes.

The limitations associated with both parametric and non-parametric models lead to an alternative hybrid model. The hybrid model is to mix the dominant genes taken from both the parametric and non-parametric models to minimise the economic cost and maximise the coverage ability. For instance, a filtered historical simulation (hereafter, FHS) combines the non-parametric density function with the GARCH filtration of the return data. It is reported that FHS generally outperforms the parametric and the non-parametric models Barone-Adesi et al. (1999, 2002); Kuester et al. (2006); Pritsker (2001). Hence, a more flexible hybrid approach is called for, in order to obtain a more accurate forecast of a time-varying non-normal return distribution.

Recently, a conditional autoregressive quantile model (hereafter CAViaR, Engle and Manganelli, 2004) utilises an empirical quantile without any assumption on the distribution for VaR modelling. However, the performance has been mixed, so far Engle and Manganelli (2004); Kuester et al. (2006).

In line with recent developments in risk modelling, this paper contributes to this growing literature by introducing a novel hybrid technique which is capable of modelling and forecasting a non-parametric density function of the asset return in a flexible manner. In the context of developing hybrid models for the density function, our study utilises the functional autoregressive modelling of the non-parametric density function (Bosq, 2000; Cardot et al., 1999, 2003, 2007; Park and Qian, 2007, 2011, hereafter FAR) to forecast a daily VaR using intraday data.[3] We develop a two-stage approach for VaR forecasting. First, we construct the sequence of empirical intraday density functions by a kernel density estimation using the high-frequency data and then we model the time-dependence of the density function by an autoregressive process in the functional space. Based on the estimated model, we can easily obtain a density forecast conditional on the past information set. This first-stage is well established in Park and Qian (2007, 2011) but it needs an additional stage to extend the intraday return density function to the daily return density function to obtain

[2]EVT draws the worst event out of the considered period (such as 1 week or 1 month) then uses them as an extreme sample.

[3]This functional approach is also in line with modelling the dynamics of economic functions recently documented by Bowsher and Meeks (2008), employing a functional signal plus noise, (FSN)-ECM, and Kargin and Onatski (2008), employing a functional autoregressive model for forecasting a yield-curve.

a daily VaR. In the second-stage, therefore, we employ the Normal Inverse Gaussian function developed by Bandorff-Nielsen (1997). While it is a parametric extension, it has the advantage of utilising the first four moments obtained from the intraday density function.

FAR approach has three important advantages in density forecasting with the VaR application. First, it models the dynamics of the non-parametric density function in the functional space. By estimating the density function non-parametrically, we can avoid uncertainties from both misspecification and the estimation errors in the parametric approach. On the other hand, the dynamics of the density function is too complicated to be specified in a parametric way. In general, all moments and the locations of the density function are contemporaneously related and their structure is time-dependent. A time-varying conditional moments and quantile approach is unable to identify their true time-dependent structure. But FAR can specify the contemporaneous and time-dependent structure in a reduced form via an autoregressive operator. It is therefore a much more attractive approach. Second, our proposed approach provides a framework for a robust VaR application, successfully minimising the economic cost and maximising the coverage ability. Existing parametric and non-parametric models are unsatisfactory in the following sense: a parametric model can minimise the economic cost but tends to underestimate the VaR (e.g., the GARCH family). A non-parametric model can provide robust and conservative coverage but fails to minimise the economic cost because the dynamics of the return distribution are neglected. The FAR approach overcomes these limitations and provides the desirable features of VaR modelling. Third, it allows us to utilise intraday information for financial risk management. The key issue is estimating and forecasting the daily risk in the investor's or bank's asset portfolio. A return is calculated daily and the volatility of the return is the key measure of risk. However, it has been documented in the market microstructure literature that the use of the high-frequency data provides a better account of the daily volatility:(Andersen et al., 2001; Bollen and Inder, 2002). Andersen and Bollerslev (1998) show how the high-frequency intraday returns contain valuable information for the measurement of volatility at the daily level. Intraday returns encompass important information which is relevant to market participants in forming their future expectations. This is especially true for large scale institutional trading with active market movements. The intraday return distribution is a key input to access the accumulated results of the daily return. The true interaction among traders happens at the intraday level in all the liquid markets. Therefore, these arguments and the increased availability of the high-frequency financial data motivate us to examine the daily return distribution using the high-frequency data. Given that high-frequency financial data is often characterised as extremely dispersed and non-normally distributed (Hasbrouck, 2007), FAR offers an ideal setting to abstract the daily volatility and the higher order moments from high-frequency data.

We demonstrate the methodological advantage of FAR through extensive evaluation schemes using both actual and simulated data. We first find that FAR is the best predictor for intraday return density forecasting, outperforming alternative functional models such as AVE (IID density functions) and LAST (martingale density function). Employing divergence criteria such as the Hilbert norm, uniform norm and entropy measure for an analysis of the thirty stock components of the DJIA and their equal weighted portfolio over 2000–2008, we find that the density forecast estimated by FAR is closest to the realised density in terms of 'closeness' in a functional space. We also conduct a Monte Carlo simulation study in which we generate the

sample based on the empirical findings, and evaluate the forecasting precision by comparing a VaR forecast with a true VaR. Employing Bias, RMSE (Root Mean Square Error), MAE (Mean Absolute Error), MAPE (Mean Absolute Percentage Error), and PMAD (Percentage Mean Absolute Deviation), we find that FAR provides the best precision of VaR forecasting for all precision measures.

Moreover, we demonstrate its advantage for VaR applications by evaluating its performance against existing popular models using a wide range of assessment criteria with 30 stocks and the equal-weighted DJIA portfolio over 2000–2008. The validation of VaR models is investigated with the set of backtesting tools. The backtesting is essential to the Basel committee's decision to allow internal VaR models for capital requirements. We adopt various backtesting tools developed in the academic literature and mentioned in the regulatory documents. Backtesting is a formal framework to verify whether the actual loss is in line with the projected loss. This involves systematically comparing the history of VaR forecasts with their associated (portfolio) returns. When a model is well specified, the number of observations falling outside VaR should be in line with the given nominal level. The number of excesses is known as the 'number of exceptions' (Jorion, 2006, p. 140). The number of exceptions exceeding the cut-off indicates that the model underestimates VaR. If so, too little capital is allocated to the risk-taking units. The regulators impose penalties on banks that adopt such an underestimated model. On the other hand, a number of exceptions lower than the cut-off indicates that the model overestimates VaR, implying that the capital allocation is inefficient. Consider the following assessment. First, coverage ability and economic cost are examined. The empirical coverage probability and the predictive quantile loss are the conventional quantitative measures. We also examine the performance by checking against the Basel Committee's recommendations such as the Basel penalty zone and the market risk capital requirement (hereafter, MRCR). The Basel Committee provides those measures from sufficient simulation studies as well as empirical analyses of financial institutions. The empirical coverage probability and the Basel penalty zone evaluate the coverage ability while the predicted quantile loss and the MRCR evaluate economic cost. Second, regulators also require that formal statistical tests should be conducted so as to evaluate the accuracy of the VaR forecast. To this end, we consider the unconditional test (Kupiec, 1995), conditional coverage tests Christoffersen (1998); Engle and Manganelli (2004) and dynamic quantile tests (Engle and Manganelli, 2004).

From the empirical evaluations of alternative VaR models, we observe that hybrid models such as FAR and FHS outperform parametric models such as GARCH, EVT and CAViaR. The hybrid models obtain high scores in all backtestings. The GARCH models tend to underestimate the VaR, as reported in Johansson et al. (1999) and Netftci (2000). They perform badly in coverage ability evaluations. They are more often rejected by all the conditional coverage tests but do well for the economic cost analysis. On the contrary, the HS performs well in the coverage ability evaluations but badly in the economic cost analysis. It is however often rejected by all the conditional coverage tests. This clearly demonstrates that there is a trade-off between the two approaches. The EVT models mostly overestimate the VaR, as expected. They perform badly for both the coverage ability evaluations and the economic cost analysis. They are often rejected by the conditional coverage tests while the filtered EVT models are relatively less rejected by the dynamic quantile tests. CAViaR is the worst performing model for all the backtestings. Therefore, the evaluation results support that the dynamic modelling of non-parametric density functions outperforms both the static

non-parametric models and the dynamic parametric models. Moreover, empirical findings that the hybrid models outperform other models are consistently repeated for the alternative positions (the short and the long position) and the alternative window sizes (of 250 and 500 business days).

The rest of this chapter is organised as follows. In Section 2, we describe the functional autoregressive modelling for VaR forecasting, along with alternative VaR models. Section 3 provides an overview of the existing backtestings in terms of quantitative measures and statistical tests. Section 4 presents the intraday data used in the analysis. Section 5 provides the Monte Carlo simulation study and the empirical evaluation of functional models and alternative VaR models using 30 components for DJIA and their equal weighted portfolio. Section 6 gives some concluding remarks.

2 VaR Models

Quantitative financial risk management is generally performed on a daily basis: an asset return is calculated daily and its volatility is measured as the core of the daily risk. Recently a better account of daily volatility by utilising high-frequency data has been well documented in the market microstructure literature (Andersen and Bollerslev, 1998; Andersen et al., 2001; Bollen and Inder, 2002). They explain that the intraday data contains valuable information which is relevant to market participants for forming their future expectations. Especially, those arguments are true for large institutional trading with active market movements. Hence, it is key to access the daily return risk to understand the behaviour of the intraday return distribution. Also, the increasing availability of high-frequency financial data motivates us to model the dynamics of the intraday return distribution first and utilise it for forecasting the daily return risk. Furthermore, given that high-frequency financial data often possesses extreme dispersion and is non-normally distributed (Hasbrouck, 2007), FAR offers an ideal way to abstract the daily volatility and higher moments from the high-frequency data. We first present our proposed model below.

2.1 Hybrid Functional Autoregressive Modelling of Non-parametric Density

We assume that trading prices are observed in a fixed interval (e.g., 5-minute)[4] and define an intraday return by the first difference of the log price,

$$r_{ti} = \ln P_{ti} - \ln P_{t,i-1},\ t = 1,\ldots,T;\ i = 1,\ldots,m \tag{1}$$

where t and i denote the tth day and its ith intraday observation, respectively. Especially, $P_{t,0}(= P_{t-1,m})$ denotes the close price yesterday, so we include overnight information in the first observation (r_{t1}) at the open time today. Further, the daily return is defined as the sum of the intraday returns:

$$r_t = \sum_{i=1}^{m} r_{ti},\ t = 1,\ldots,T. \tag{2}$$

[4]In practice, the trading action is not regular in time so that the frequency is generally irregular. We thus filter observations and draw prices in a fixed time base, e.g., 9:35 AM, 9:40 AM, ..., 4:00 PM.

In sum, we have the sequence of time series such that

$$X = (X_t)_{t=1}^{T} \text{ and } X_t = (r_{ti})_{i=1}^{m}, \tag{3}$$

where X is the set of intraday return paths and X_t is the intraday return path at the tth day.

First, we assume that the intraday return series, $r_{ti} \in X_t$, is strictly stationary within a given day and the intraday density function can be estimated for each time point t (day) given the sample observation. Once we denote the intraday density function by f_t, the sequence $(f_t)_{t=1}^{T}$ can be defined in the functional space. Second, we assume that this local stationarity does not carry over to a longer horizon. This implies that the time series $(X_t \in X)$ is nonstationary, that is, the intraday density function is time-varying. These assumptions are known as the "piecewise stationarity" of a stochastic process.[5] In the general framework, "piece" is taken as any appropriate time unit such as a day, week or month. Thus a day is a "piece" in our study. FAR has been developed based on the "piecewise stationarity" condition for the observed stock return series. The intraday return series is stationary in a day but it will be nonstationary in the long-run, that is, each day would have a different intraday return distribution.

We assume that the time-varying intraday density function (f_t) follows an autoregressive process with lag order one[6]

$$w_t = Aw_{t-1} + \varepsilon_t,\ t = 2, \ldots, T, \tag{4}$$

where $w_t = f_t - \mathbb{E}f$ is the fluctuation of the density function from the well-defined common expectation of the density function $(\mathbb{E}f)$, A is an autoregressive operator on the Hilbert space $(\mathscr{H})$ satisfying $\|A\| < 1$, and $(\varepsilon_t)_{t=1}^{T}$ is the sequence of the functional white noise process. Eq. (4) can be rewritten for the density function:

$$f_t = \mathbb{E}f + Aw_{t-1} + \varepsilon_t,\ t = 2, \ldots, T. \tag{5}$$

Thus the one-step ahead forecast of the density function is obtained by the conditional expectation on the past information set $(\mathscr{F}_{t-1})$:

$$\mathbb{E}\left[f_t | \mathscr{F}_{t-1}\right] = \mathbb{E}f + Aw_{t-1}. \tag{6}$$

FAR is a very general modelling of the conditional autoregressive moments. For example, the centred mean in (4), $\tilde{\mu}_t = \langle x, w_t \rangle$, can represented by

$$\tilde{\mu}_t = \langle x, Aw_{t-1} \rangle + \langle x, \varepsilon_t \rangle = \langle A^* x, w_{t-1} \rangle + \eta_t, \tag{7}$$

where A^* is the adjoint of A, $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathscr{H}$, $\langle v, u \rangle = \int_{\mathbb{R}} vu(u)\,du$, and $(\eta_t)_{t=1}^{T}$ becomes the white noise process under the regular assumptions. The right hand side of (7) can be represented by the infinite sum of polynomials, $\sum_{k=1}^{\infty} c_k \left\langle x^k, w_{t-1} \right\rangle = \sum_{k=1}^{\infty} c_k \tilde{\mu}_{k,t-1}$, where $\tilde{\mu}_{k,t-1}$ is the first-lagged centered k-th order moment. This representation implies that the centred mean is specified by the linear com-

[5] Piecewise stationarity is well demonstrated by "Short-Time Fourier Transform" (Gabor, 1946) in the signal processing literature. The powerful spectral analysis, invented for stationary process, can be applied to nonstationary signals under this assumption.

[6] It is trivial to extend the model to longer lag orders such as an $AR(p)$ model in a vector space. Furthermore, the higher lag orders should be available for higher order FAR modelling.

bination of all the first-lagged higher order moments. Hence, it includes ARCH-M as a very specific case. Moreover, the centred second order moment, $\tilde{\mu}_{2t} = \langle x^2, w_t \rangle$, is analogously expressed by all the first-lagged higher moments. It is thus the generalised conditional autoregressive process of the second order moment. These special cases appear in Park and Qian (2007) who derive ARCH and ARCH-M from FAR as special cases. Furthermore, a conditional autoregressive quantile process could be represented as a special case of FAR too. Intuitively, (5) can be applied to the quantile process. Let q_t satisfy $\alpha = \int_{-\infty}^{q_t} f_t(s)\,ds$. Then the quantile process (q_t) tends to revert to the α-quantile of $\mathbb{E}f\ (\bar{q})$, as long as the mean reverting property holds in (5). This can be modelled by two prats; $\bar{q}$ and the adjustment of the deviation between q_{t-1} and $\bar{q}$. Hence, it follows that $q_t = g(\bar{q}, q_{t-1} - \bar{q}) + \xi_t$, where g is the general form of the nonlinear function. Even though the nonlinear functional form is implicit for the conditional quantile process, the autoregressive adjustment is true as long as FAR holds. Consequently, the ARCH process and the conditional quantile process can be regarded as special cases under some restrictions on the structure of the time-dependence. Usually those parametric models suffer from a misspecification of the dynamic structure when their underlying assumptions are not valid. On the contrary, FAR represents general conditional autoregressive moments for all higher orders and conditional quantile process by the autoregressive operator which is the reduced form for all such conditional autoregressive process. Therefore, FAR can be more flexible and improve the modelling of a time-varying asset return distribution.

Moreover, FAR reasonably explains the finance theories and findings of asset returns. Consider that the mean of the return obtained from f_t is constant and equivalent to that from $\mathbb{E}f$ in (4). This implies that there is no expected excess returns since $\langle x, w_t \rangle = \langle x, w_{t-1} \rangle = 0$ in (4). Hence, FAR can present the weak-form efficiency in the 'Efficient Market Hypothesis' with time-varying higher order moments such as the ARCH process. If the mean of the return has a long-memory, the mean would described by the AR process. Hence, FAR supports 'behavioural finance' arguments such as overconfidence, overreaction, and momentum (gradual adjustment to a new equilibrium) with time-varying higher order moments.

If A is the zero operator, the density function of the intraday return is independently and identically distributed in the functional space.

$$f_t = \mathbb{E}f + \varepsilon_t. \tag{8}$$

The best predictor is therefore the common expectation of the density function. Since the expected return is equivalent to the average return obtained from the $\mathbb{E}f$ and higher order moments are time constant, this model implies the 'Efficient Market Hypothesis' with time constant higher order moments. We denote this model by AVE.

If A the identity operator, the density function follows the functional martingale process:

$$f_t = f_{t-1} + \varepsilon_t. \tag{9}$$

The last observation is therefore the best predictor. All conditional moments including the expected return are nonstationary. Hence, any theories and arguments from the finance literature cannot be applicable to this case. It may be locally observed under the circumstance of "Black Swan" such as the Wall Street Crash (1929), Black Monday (1987) and the Financial Crisis (2007), since the shock used to be continued or

diverge in the short term.

To apply FAR in practice, we first need to estimate the empirical intraday density functions at each point of time t. To this end, we suggest using the kernel density estimation. A standard kernel estimator is typically defined by

$$\hat{f}_t(x) = \frac{1}{mh_t}\sum_{i=1}^{m} K\left(\frac{x - r_{it}}{h_t}\right),\ t = 1, \ldots, T \tag{10}$$

where K is a kernel, m is the number of observations and h_t is a bandwidth (smoothing parameter or window width). One practically important issue is the selection of an appropriate kernel and the bandwidth (usually selected by cross-validation criteria). We follow Silverman (1986) and use a Gaussian kernel with an optimal bandwidth given by $1.06\hat{\sigma}_t m^{-1/5}$ where $\hat{\sigma}_t$ is the sample standard deviation of r_{it}.[7] Given the sequence of the estimated density functions, $\left(\hat{f}_t\right)_{t=1}^{T}$, we estimate $\mathbb{E}f$ by the sample average of $\hat{f}_t$ such that $\bar{f} = T^{-1}\sum_{t=1}^{T}\hat{f}_t$. Then the sequence of the fluctuation $(\hat{w}_t)_{t=1}^{T}$ is obtained by $\hat{w}_t = \hat{f}_t - \bar{f}$.

We then estimate the autoregressive operator (A) which can be obtained by utilising the autocovariance operators of order 0 (C_0) and 1 (C_1) in infinite dimensions,

$$C_s = \mathbb{E}\left(w_t \otimes w_{t-s}\right),\ s = 0, 1,$$

where $\otimes$ denotes a tensor product in the infinite dimensional space.[8] Using the relationship $C_1 = AC_0$, we obtain an autoregressive operator of order one:

$$A = C_0^{-1}C_1. \tag{11}$$

Since autocovariance operators are consistently estimated by $\hat{C}_s = (T-1)^{-1}\sum_{t=2}^{T}(\hat{w}_t \otimes \hat{w}_{t-s})$ for $s = 0, 1$, we can consistently estimate the autoregressive operator of order one by $\hat{A} = \hat{C}_0^{-1}\hat{C}_1$. Using the spectral representation for a compact and self-adjoint operator C_0,

$$C_0 = \sum_{\ell=1}^{\infty}\lambda_\ell\left(v_\ell \otimes v_\ell\right), \tag{12}$$

where (λ_ℓ, v_ℓ) are the pair of eigenvalue and eigenfunction of C_0, the inverse of C_0 can be easily obtained by

$$C_0^{-1} = \sum_{\ell=1}^{\infty}\lambda_\ell^{-1}\left(v_\ell \otimes v_\ell\right). \tag{13}$$

Since C_0 is defined on an infinite dimensional space in principle, there is an ill-posed inverse problem. In other words, it needs an infinite number of eigenvalues and their corresponding eigenfunctions. To avoid this ill-posed inverse problem, we project A on a finite subspace of $\mathscr{H}$, define V_L as the subspace of $\mathscr{H}$ spanned by the L-eigenfunction, $v_1, \ldots, v_L$, and let $C_{0,L} = \Pi_L C_0 \Pi_L$, where Π_L is the projector on V_L.[9] Then

[7] Various other kernels are also available in the literature including Epanechnikov, Bi-weight, Triangular and Rectangular.

[8] It is defined as $(u \otimes v) = \langle v, \cdot\rangle u$ which is equivalent to the outer product uv' in a finite dimensional vector space.

[9] In practice, the choice of L is guided by applying a functional principle component analysis (FPCA) and a cross validation (CV) method. FPCA explains the variation of the fluctuation and CV chooses an optimal dimension $L\,(\leq L_{max})$ by minimising the

the inverse of C_0 is approximated by

$$C_{0,L}^{+} = \sum_{\ell=1}^{L} \lambda_{\ell}^{-1} \left(v_{\ell} \otimes v_{\ell} \right), \tag{15}$$

which is defined on V_L. Hence, the estimator of the autoregressive operator on the subspaceV_L of $\mathscr{H}$ is estimated by

$$\hat{A}_L = \hat{C}_{0,L}^{+} \hat{C}_1. \tag{16}$$

Under some regularity conditions, Park and Qian (2007, 2011, Theorem 5) show that $\hat{A}_L$ is a consistent estimator (See also Bosq, 2000; Cardot et al., 1999, 2003, 2007) and that the forecast errors follow the normal distribution asymptotically (see Theorem 8 in Park and Qian, 2007):

$$\left\| \hat{A}_L - A \right\| \xrightarrow{a.s.} 0 \text{ and } \sqrt{T/L} \left(\hat{A}_L \hat{w}_T - A w_T \right) \xrightarrow{d} \mathbb{N}\left(0, \Sigma\right),$$

where L satisfies the condition $LT^{-1/4} \log T \to 0$ requiring that L should not increase too fast with T and Σ is the covariance operator of ε_t, $\mathbb{E}\left(\varepsilon_t \otimes \varepsilon_t\right)$.

The one-step ahead forecast of the density function is therefore evaluated by

$$\hat{f}_{T+1} = \bar{f} + \hat{A}_L \hat{w}_T. \tag{17}$$

Figure 1 illustrates the forecasting mechanism of FAR using the equal weighted DJIA portfolio sample. Panel (a) and (b) present the density forecast and its forecasting error. Panel (c) shows that the fluctuation gradually disappears over time. Equivalently, Panel (d) shows that the forecasting mechanism of the FAR is the mean reversion of the density function.

[FIGURE 1 ABOUT HERE]

To reduce the required computing time, $\hat{w}_t$ is often approximated by the Fast Fourier Transformation or the Wavelet transformation. Those approximations reduce the computing time substantially through shrinking the dimension of the function.[10] Second, they produce accurate approximations of the tail behaviour so that the VaR estimates can be improved. After obtaining the forecast of the transformed $\hat{w}_{T+1}$, the function is finally inverted to the original one Antoniadisa and Sapatinas (2003); Besse et al. (2000); Lee and Ready (1991).[11]

For the purpose of the daily VaR forecast using the intraday density forecast, we first transform the intraday density function to the daily density function by deriving the first four moments of the intraday density

following criterion:

$$\sum_{i=1}^{N_{cv}} \left\| \hat{w}_{T-i+1}^{L} - \hat{w}_{T-i+1} \right\|^2 = \sum_{i=1}^{N_{cv}} \int \left[\hat{w}_{T-i+1}^{L}(x) - \hat{w}_{T-i+1}(x) \right]^2 dx, \tag{14}$$

where N_{cv} is the number of the last observations used in CV and $\hat{w}_{T-i+1}^{L}$ are the in-sample forecasts of w_{T-i+1} on L-dimensional subspace. We set $L_{max} = 20$ through our study and find that the cross-validation procedure given by (14) selects the optimal value of L ranging between 3 and 10.

[10] It is usually thirty times faster than the non-transformed case in our study.

[11] We employ the Fast Fourier Transformation in the empirical evaluation below. The Wavelet transformation provides quantitatively similar results, which are available upon request.

function and then construct the daily density function by the Normal Inverse Gaussian (NIG) distribution that is one of the most popular distribution families proposed in the literature (Bandorff-Nielsen, 1997) for describing return distributions. Let μ_t, v_t, s_t, and k_t be, respectively, the mean, variance, skewness, and kurtosis of the intraday density function. Then we calculate four parameters, $(\alpha_t, \beta_t, \gamma_t, \delta_t)$, that determine the shape of the NIG distribution using the following formulas

$$\alpha_t = v_t^{-\frac{1}{2}} \left(3k_t - 4s_t^2 - 9\right)^{\frac{1}{2}} \left(k_t - \frac{5}{3}s_t^2 - 3\right)^{-1}, \ \beta_t = s_t v_t^{-\frac{1}{2}} \left(k_t - \frac{5}{3}s_t^2 - 3\right)^{-1},$$
$$\gamma_t = \mu_t - 3s_t v_t^{\frac{1}{2}} \left(3k_t - 4s_t^2 - 9\right)^{-1}, \ \delta_t = 3^{\frac{3}{2}} \left\{ v_t \left(k_t - \frac{5}{3}s_t^2 - 3\right)\right\}^{\frac{1}{2}} \left(3k_t - 4s_t^2 - 9\right)^{-1}, \tag{18}$$

where the kurtosis should satisfy $k_t > 3 + (5/3)\, s_t^2$. Furthermore, α_t determines the tail heaviness, β_t the asymmetry, γ_t the location and δ_t the scale of the distribution. We assume that the intraday returns are strictly stationary on the tth day and the intraday density function can be consistently approximated by the following NIG density function:

$$g_{,t}(x) = \frac{\alpha_t \delta_t J_1 \left(\alpha_t \sqrt{\delta_t^2 + (x-\gamma_t)^2}\right)}{\pi \sqrt{\delta_t^2 + (x-\gamma_t)^2}} \exp\left[\delta_t \lambda_t + \beta_t (x - \gamma_t)\right], \tag{19}$$

where $\lambda_t = \sqrt{\alpha_t^2 - \beta_t^2}$ and J_1 denotes the modified Bessel function of the second kind. Under the strict stationarity assumption for the intraday return, the density function of the daily return, $r_t \left(= \sum_{i=1}^{m} r_{it}\right)$, can be consistently approximated by the NIG density function such that

$$r_t \sim \mathbb{NIG}\left(\alpha_t, \beta_t, m\gamma_t, m\delta_t\right), \tag{20}$$

where m is the number of intraday return observations.[12] We then finally obtain the daily VaR from the inverse cumulative density function of NIG distribution:

$$VaR_t(\alpha) = G_t^{-1}(\alpha), \tag{21}$$

where $G(\cdot)$ denotes the cumulative density function of the NIG distribution.

In practice, we first calculate the first fourth empirical moments $\hat{\mu}_{T+1}$, $\hat{v}_{T+1}$, $\hat{s}_{T+1}$ and $\hat{k}_{T+1}$ by numerical integration of the intraday density forecast, $\hat{f}_{T+1}$. Then, the four moments are utilised to obtain a daily VaR forecast following the procedures (18)–(21).

In sum, FAR utilises a non-parametric density function, which overcomes many of the limitations of the parametric approach. Furthermore, imposing an autoregressive structure on the non-parametric density function improves the defect of the static non-parametric approach. Hence, it is expected that FAR can

[12] The NIG distribution is close under convolution for independent random variables X and Y

$$X \sim \mathbb{NIG}(\alpha, \beta, \mu_X, \delta_X), Y \sim \mathbb{NIG}(\alpha, \beta, \mu_Y, \delta_Y) \Rightarrow X + Y \sim \mathbb{NIG}(\alpha, \beta, \mu_X + \mu_Y, \delta_X + \delta_Y).$$

reduce the economic cost and improve accuracy in VaR analysis. Furthermore, it enjoys the rich information of the high-frequency intraday returns which are helpful in forecasting the daily risk. Therefore, FAR would be one of the most efficient and accurate models for VaR analysis.

2.2 Alternative Existing Models

We now overview popular VaR models developed by practitioners and academics including HS, FHS, GARCH models, EVT models, and CAViaR.

HS requires no explicit assumption on the asset return distribution. It non-parametrically estimates the empirical distribution from observations given the rolling window that generally ranges from 6 months to two years. Returns in the given window are sorted in ascending order and the α-quantile of interest is given by the return that leaves α percent of the observation on its left side. The rolling approach however implicitly assumes that the distribution does not change within each window. Hence, the empirical quantile estimator is consistent only if the window size goes to infinity under the time-variant condition for the distribution. In practice, the length of the window must satisfy two contradictory properties: it must be large enough in order to make statistical inference significant while it must not be too large to avoid the risk of taking observations outside of the current volatility cluster. VaR estimates based on the HS will underestimate (overestimate) the true VaR when the underlying state is moving from a low (high) to a high (low) volatility cluster. In addition, VaR estimates based on HS may present predictable jumps due to the discreteness of the extreme values.[13]

In order to mitigate the drawbacks of the HS, FHS was introduced Barone-Adesi et al. (1999, 2002). FHS first standardises the returns using the estimated conditional mean and standard deviation obtained from GARCH models.[14] Then it applies HS to the filtered returns which are independently and identically distributed.[15] Hence, FHS is a kind of hybrid non-parametric model combining HS with GARCH. By absorbing the advantages of both HS and GARCH, it improves the performance of VaR forecasting Barone-Adesi et al. (1999, 2002); Kuester et al. (2006); Pritsker (2001).

GARCH models simultaneously model a time-varying conditional mean and variance. We consider two GARCH models with differences in the distribution of the error term and the estimation of the first two conditional moments: the RM and GARCH with Student's t-distribution. The RM describes the asset return by the normal distribution. The major criticism of the RM is its symmetric distribution assumption for all asset returns and the lack of a description for the fat-tails of financial time series. Johansson et al. (1999) find that the RM significantly underestimates the true risk exposures for small and undiversified portfolios. Assuming the normal distribution when its true distribution has a fat-tail will also underestimate the risk of extreme losses (Netftci, 2000). Thus GARCH utilises Student's t-distribution, possessing a fatter tail than

[13] For example, let the window size be 180 days and suppose there is an extreme observation today. It is easy to predict that VaR jumps upward. The reverse effect will appear after 180 days, when the large observation will drop out of the window.

[14] Even if the underlying distribution is far from Gaussian, a quasi maximum likelihood estimator may be consistent and asymptotically normal (Bollerslev and Wooldridge, 1992; Lee and Hansen, 1994; Lumsdaine, 1994).

[15] Barone-Adesi et al. (1999, 2002) and Pritsker (2001) compute VaR from simulated sample paths using draws from the filtered returns. Alternatively, our paper computes VaR by applying the HS to the filtered returns without generating sample paths. The final VaR is computed by $VaR_t(\alpha) = \mu_t + \sigma_t VaR_t^*(\alpha)$, where $VaR_t^*(\alpha)$ is the VaR estimate of the filtered returns.

that of normal distribution.[16]

EVT models consider probabilities associated with extreme events such as minimum or maximum values given a block size. It is useful for modelling crashes or stress on financial assets (see Embrechts et al. (1997) for a comprehensive review of EVT models). However, it tends to overestimate the VaR in a normal period and faces many challenges including the limited number of extreme data,[17] the choice of parameter estimation and the dynamic specification of the time-dependence on those parameters. Since the distribution of extreme values is not demonstrated by the central limit theorem, we employ non-standard distribution models: the generalised extreme value (GEV) distribution and generalised Pareto distribution (GPD). Their parameters are estimated by the maximum likelihood estimator (MLE) using extreme observations. The estimation is frequently limited by the small number of extreme observations. Filtered EVT (FEVT) models are suggested to control for time-varying volatility Diebold et al. (1998); McNeil (2000). Analogous to the FHS, it first filters the returns by the GARCH estimates, then applies the standard EVT procedure to the filtered returns. The filtered versions of GEV (FGEV) and GPD (FGPD) are also considered in our paper.

Engle and Manganelli (2004) introduce CAViaR that directly models the evolution of the quantile over time instead of modelling the evolution of the asset return distributions. The advantage of CAViaR is that it relaxes the assumption on the underlying return distribution. Thus it reduces the risk from a misspecification of the parametric distribution.

According to a recent study of alternative VaR strategies Kuester et al. (2006), the FEVT models and the FHS outperform GARCH and CAViaR. The inadequate performance of GARCH and CAViaR could be attributed to a misspecification in the time-dependence of the distribution's parameters and quantiles as well as the choice of the distribution family. On the other hand, the outperformance of the filtered approaches associated to the FHS and the FEVT models suggests the advantages of a hybrid approach. In this context, we expect FAR to enjoy a similar advantage since it models the time-varying density function in the functional space without any parametric specifications on the distribution and the autoregressive operator contains the complicated contemporaneous and time-dependent structure on the moments and locations of the distribution in a reduced form. The models considered are summarised in Table 1.

[TABLE 1 ABOUT HERE]

3 Backtestings

In this section, we summarise the backtesting procedures, which are organised into two general headings: quantitative and statistical. The former intuitively evaluate the accuracy of the VaR models whilst the latter provide the formal statistical tests for both unconditional and conditional accuracy.

Before unfolding the backtesting procedures, we preliminarily define a violation of a VaR forecast.

[16] Recently, the challenge has been how to allow for conditional higher order moments in the GARCH framework, in order to improve the accuracy of VaR forecasting (Bali et al., 2008).

[17] Since minimum and maximum values are draws in a fixed block size, i.e., 100 days or 180 days, it requires sufficiently large a sample of observations to obtain the relevant size of minimum or maximum values. Hence, there is always a trade-off between the level of extremeness and the number of observations in practice.

Given the nominal coverage probability, α, a violation is defined by the indicator function

$$H_s = \mathbf{1}\left\{r_s < \widehat{VaR}_s(\alpha)\right\},\ s = 1,\ldots,N, \tag{22}$$

where $\widehat{VaR}_s(\alpha)$ is the VaR forecast given the information set available at $s-1$ $(\mathscr{F}_{s-1})$ with the nominal coverage probability, α. An accurate VaR forecast thus satisfies

$$\mathbb{E}\left[H_s|\mathscr{F}_{s-1}\right] = \alpha, \tag{23}$$

which implies that H_s is independent of any function of the variables in $\mathscr{F}_{s-1}$. Hence, accuracy implies that the conditional binomial process (H_s) on the information set available at $s-1$ should follow the IID Bernoulli distribution (Lemma 1; Christoffersen (1998)):

$$H_s|\mathscr{F}_{s-1} \sim iid\ Bernoulli(\alpha), \tag{24}$$

where $Var(H_s|\mathscr{F}_{s-1}) = \alpha(1-\alpha)$.

3.1 Quantitative Evaluations

The empirical conditional coverage probability is calculated by the sample average of the N violations of H_s, that is $\hat{\alpha} = N^{-1}\sum_{s=1}^{N} H_s$ which is the consistent estimator of the conditional coverage probability under the true forecasting model assumption. Hence, the regulator prefers a VaR model with a empirical conditional coverage probability that is closest to its nominal value. Since the empirical conditional coverage probability is random, its significance needs to be tested through formal statistical tests which will be introduced in the next subsection.

The Basel penalty zone is found in Basel Committee on Banking and Supervision (1996). It describes the strength of an internal model through the test of failure rate, which records the number of daily violations of the 99 percent VaR in the previous 250 business days. One may expect, on average, 2.5 violations out of the previous 250 VaR forecasts under the correct forecasting model. The Basel Committee rules that up to four violations are acceptable for banks and defines the range as a "Green" zone. If the number of violations is five or more, the banks fall into a "Yellow" (5–9) or "Red" (10+) zone, where the penalty is cumulatively imposed on the bank by the multiplicative factor (κ) from 3 to 4.[18] It is used for calculating the market risk capital requirement. If a bank falls into "Red" zone, the penalty is automatically generated. Whereas, if a bank is in "Yellow", the supervisor will decide the penalty depending on the reason for the violation. Jorion (2006, p. 149) summarises the categories for the reasons suggested by the Basel Committee.[19]

The MRCR (market risk capital requirement) originated from the Basel II Accord. According to the applications of the Basel II Accord, it requires the minimum capital requirement for market risk as well as credit risk and operating risk. The Basel II Accord provides two approaches to measure the MRCR: (i) Stan-

[18]The multiplicative factor corresponds to the number of violation: 0–4 (3), 5 (3.4), 6 (3.5), 7 (3.65), 8 (3.75), 9 (3.85) and 10 (4), respectively.

[19]There are four categories: (i) Basic integrity of the model, (ii) Model accuracy could be improved, (iii) Intraday trading, (iv) Bad luck.

dardised approach and (ii) Internal models approach. We choose the second approach to evaluate the VaR forecast. Providing that a bank satisfies the qualitative requirements (the bank has a sound risk management system and an independent risk-control unit as well as external audits), the MRCR is summarised by the following four factors: (i) quantitative parameters, (ii) treatment of correlations, (iii) market risk charge, and (iv) plus factor.[20] The MRCR thus can be formulated by

$$MRCR_s = \max\left(\kappa\frac{1}{60}\sum_{i=1}^{60} VaR_{s-i}(\alpha), VaR_{s-1}(\alpha)\right) + SRC_s,\ s = 251,\ldots,N, \tag{25}$$

where SRC is the additional capital charge for the specific risk Basel Committee on Banking and Supervision (1996, 2004) but it is ignored in our evaluation, since it is assumed to be a common value applied to all VaR models. κ is the Basel penalty factor. A model carrying the minimum MRCR is preferred by the regulator.

Predictive quantile loss evaluates a VaR forecast in terms of economic cost. We employ the predictive quantile loss using the "check" function of Koenker and Bassett (1978) which can be regarded as a "predictive" quasi-likelihood (Bertail et al., 2004; Komunjer, 2004). The expected loss of VaR for a given α is

$$Q_s(\alpha) = \mathbb{E}\left[(\alpha - H_s)\left(r_s - \widehat{VaR}_s(\alpha)\right)\right], \tag{26}$$

which can provide the measure of the lack-of-fit for a quantile model and be interpreted as the economic cost to carry a VaR model. The expected check function, $Q(\alpha)$, is estimated from the sequence of N violation indicates (H_s):

$$\hat{Q}(\alpha) = \frac{1}{N}\sum_{s=1}^{N}(\alpha - H_s)\left(r_s - \widehat{VaR}_s(\alpha)\right). \tag{27}$$

A model providing the minimum value of $\hat{Q}(\alpha)$ is preferable from the regulator's point of view.

3.2 Statistical Evaluations

The unconditional coverage test was first developed by Kupiec (1995). If we assume that VaR forecasts are independent over time, the violations of a VaR forecast can be regarded as the realisation of an independent binomial random variable whose probability of a realised return's exceeding the corresponding VaR forecast is equal to the nominal coverage probability, α. Therefore, accurate VaR forecasts should satisfy the requirement that their empirical coverage probability equals the nominal coverage probability. The test thus describes the null and alternative hypotheses by

$$\mathbb{H}_0 : \mathbb{E}[H_s] = \alpha \text{ against } \mathbb{H}_1 : \mathbb{E}[H_s] \neq \alpha. \tag{28}$$

Under the null hypothesis, (24) can be employed for constructing the likelihood ratio test statistic

$$LR_{uc} = 2\left[\ell(\hat{\alpha}; H_1,\ldots,H_N) - \ell(\alpha; H_1,\ldots,H_N)\right] \overset{d}{\sim} \chi_1^2, \tag{29}$$

[20] See Basel Committee on Banking and Supervision (1996, 2004) for details and Jorion (2006) for a compact summary.

where $\ell(\cdot)$ denotes the log-likelihood for the Bernoulli distribution such that

$$\ell(\hat{\alpha}; H_1, \ldots, H_N) = N_1 \log \hat{\alpha} + N_0 \log(1-\hat{\alpha}), \tag{30}$$

$$\ell(\alpha; H_1, \ldots, H_N) = N_1 \log \alpha + N_0 \log(1-\alpha). \tag{31}$$

The MLE of $\hat{\alpha}$ is the ratio of the number of violation, N_1, to the total number of observation, $N_0 + N_1 = N$, that is, $\hat{\alpha} = N_1/(N_0+N_1)$. Kupiec (1995) however reports that this test has a low power to distinguish among alternative hypotheses, even in sufficiently large samples. Hence, the regulator would have the low confidence in the bank's VaR report when its internal model can not be rejected by this test.

In the presence of time-dependence, volatility clustering and persistence are often found in financial time series. Therefore, the conditional accuracy of VaR forecasts is highly important. If a VaR model ignores such underlying state, its VaR estimates may have an incorrect conditional coverage probability whilst its unconditional coverage probability is still correct.[21] The regulator thus requires that the VaR model exhibit an accurate coverage probability regardless of the underlying state. In order to test the conditional accuracy of a VaR forecast, Christoffersen (1998) has developed the conditional coverage test by combining the unconditional coverage test and the independence test: resulting log-likelihood ratio test statistic is the summation of two log likelihood ratio test statistics:

$$LR_{cc} = LR_{uc} + LR_{ind} \overset{d}{\sim} \chi^2_2, \tag{32}$$

where LR_{uc} and LR_{inc} are the log likelihood ratio test statistics of the unconditional coverage test and the independence test, respectively. The independence test utilises the sequence of violation indicates (H_s) as a binary first-order Markov chain with transition probability matrix,

$$\Pi = \begin{pmatrix} \pi_{00} & \pi_{01} \\ \pi_{10} & \pi_{11} \end{pmatrix}, \tag{33}$$

where $\pi_{ij} = \mathbb{P}(H_s = j | H_{s-1} = i)$ for $i, j = 0, 1$. Under the null hypothesis of independence, the likelihood ratio test statistic is given by

$$LR_{ind} = 2\left[\ell\left(\hat{\Pi}; H_2, \ldots, H_N | H_1\right) - \ell\left(\hat{\pi}_1; H_2, \ldots, H_N | H_1\right)\right] \overset{d}{\sim} \chi^2_1. \tag{34}$$

The first joint log likelihood conditional on the first observation is

$$\ell(\Pi; H_2, \ldots, H_N | H_1) = N_{00} \log \pi_{00} + N_{01} \log \pi_{01} + N_{10} \log \pi_{11} + N_{11} \log \pi_{11}, \tag{35}$$

where N_{ij} denotes the number of transitions from state i to j, $N_{ij} = \sum_{s=2}^{N} \mathbf{1}(H_s = j | H_{s-1} = i)$, and the MLEs of π_{01} and π_{11} under the alternative hypothesis are $\hat{\pi}_{01} = N_{01}(N_{00}+N_{01})^{-1}$ and $\hat{\pi}_{11} = N_{11}(N_{10}+N_{11})^{-1}$, respectively. Under the null hypothesis of independence, we have $\pi_{01} = \pi_{11} \equiv \pi_1$, from which we obtain the

[21] An HS ignores the time-dependent structure of the underlying probability law of financial time series. It is frequently rejected for the independence test whilst it provides an accurate unconditional coverage by the unconditional coverage test.

second conditional log likelihood as follows:

$$\ell(\pi_1; H_2, \ldots, H_N | H_1) = (N_{00} + N_{10}) \log(1 - \pi_1) + (N_{01} + N_{11}) \log \pi_1, \tag{36}$$

where the MLE of π_1 is $\hat{\pi}_1 = (N_{01} + N_{11})(N_{00} + N_{10} + N_{01} + N_{11})^{-1}$. Hence, the LR test statistic in (32) is alternatively expressed by combining (29) and (34):

$$LR_{cc} = 2\left[\ell\left(\hat{\Pi}; H_2, \ldots, H_N | H_1\right) - \ell(\alpha; H_2, \ldots, H_N | H_1)\right] \overset{d}{\sim} \chi^2_2, \tag{37}$$

where the second log likelihood conditional on the first observation is

$$\ell(\alpha; H_2, \ldots, H_N | H_1) = (N_{00} + N_{10}) \log(1 - \alpha) + (N_{01} + N_{11}) \log \alpha. \tag{38}$$

Therefore, the regulator requires that the accuracy of VaR forecasts reported by a bank's internal model should be evaluated by the conditional coverage test rather than the single unconditional coverage test or the independence test. If the VaR model is rejected by the conditional coverage test, the regulator would recommend that the bank improve its VaR model to eliminate the time-dependent coverage ability in VaR forecasts. The proposed test however has a limitation, since it allows only the time-dependence of the first lag-order in the sequence of violation indicators (Gaglianone et al., 2009). The testing procedure may be extended by allowing more lag orders.

The dynamic quantile test (Engle and Manganelli, 2004) is a general extension of the conditional coverage test allowing for more time-dependent information of $(H_s)_{s=1}^N$. It regresses H_s on some carefully selected explanatory variables in $\mathscr{F}_{s-1}$ such that

$$H_s = \alpha_0 + \sum_{i=1}^{p} \beta_i H_{s-i} + \beta_{p+1} \widehat{VaR}_s(\alpha) + u_s,\ s = p+1, \ldots, N. \tag{39}$$

Under the null hypothesis, the explanatory variables should have no power on H_s, that is

$$\mathbb{H}_0 : \alpha_0 = \alpha \text{ and } \beta_i = 0 \text{ for } i = 1, \ldots, p+1. \tag{40}$$

By subtracting α from both sides of (39), we have

$$(H_s - \alpha) = (\alpha_0 - \alpha) + \sum_{i=1}^{p} \beta_i H_{s-i} + \beta_{p+1} \widehat{VaR}_s(\alpha) + u_s,\ s = p+1, \ldots, N. \tag{41}$$

The simple vector notation of (41) is given by

$$y_s = Z_s \beta + u_s,\ s = p+1, \ldots, N, \tag{42}$$

where $y_s = H_s - \alpha$, $Z_s = \left(1, H_{s-1}, \ldots, H_{s-p}, \widehat{VaR}_s(\alpha)\right)'$ and $\beta = (\alpha_0 - \alpha, \beta_1, \ldots, \beta_{p+1})$. We can therefore express the null hypothesis in (40) as

$$\mathbb{H}_0 : \beta = 0. \tag{43}$$

Under the null hypothesis, the least squares estimator converges to the normal distribution such that

$$\hat{\beta} = \left(\mathbf{Z}'\mathbf{Z}\right)^{-1}\mathbf{Z}'\mathbf{y} = \left(\mathbf{Z}'\mathbf{Z}\right)^{-1}\mathbf{Z}'\left(\mathbf{H} - \alpha\iota\right) \overset{d}{\sim} N\left(\mathbf{0}, \left(\mathbf{Z}'\mathbf{Z}\right)^{-1}\alpha\left(1-\alpha\right)\right). \tag{44}$$

Then we can deduce the test statistics from Engle and Manganelli (2004) by

$$DQ = \frac{\hat{\beta}'\mathbf{Z}'\mathbf{Z}\hat{\beta}}{\alpha\left(1-\alpha\right)} \overset{d}{\sim} \chi^2_{p+2}. \tag{45}$$

The Monte Carlo simulation study by Berkowitz et al. (2011) shows that the dynamic quantile test appears to be the best backtest for a 99% VaR whilst other tests generally have much lower power against incorrect VaR models.[22]

In summary, regulators require banks to evaluate the accuracy of VaR forecasts by the bank's internal models using statistical tests. There are two popular tests: the conditional coverage test Christoffersen (1998) and the dynamic quantile tests (Engle and Manganelli, 2004). The VaR models in our study are therefore mainly evaluated by these two conditional tests. Since VaR forecasts are generally evaluated in very small coverage probability (1%), testing power is usually less than those for the central statistics such as mean or variance. Hence it would be undesirable to evaluate the VaR model at a lower significance level (e.g., 1%). The regulator should place more confidence in VaR models not rejected by the conditional coverage test at a more restrictive significance level.

4 Data

We choose the 30 companies of the DJIA as the sample of this study. This Wall Street benchmark index contains the 30 largest stocks listed in the US market. These corporations are deemed to be too big to fail even during the financial crisis. Among these companies, three of them were rescued by the U.S. government in 2008.[23] Understanding the risk profiles of these 30 companies is essential for monitoring and safeguarding the financial system. Furthermore, unlike small companies, these companies are actively traded in a trading section which generates enormous amount of information. It is an interesting empirical question to ask whether the financial market revealed any insight regarding these company's risks during the intraday trading.

The list of components changes annually, and we choose the list in year 2005, because this year is in the middle of our whole sample period, which is from 2000 to 2008. Intraday transaction data are collected from Trade and Quote (TAQ) dataset. Following traditional studies which deal with intraday data, such as Lee and Ready (1991) and Hvidkjaer (2006), a filtering procedure has been applied to exclude any data likely to

[22] Berkowitz et al. (2011) make the criticisism that estimating the autoregression (42) by ordinary least squares can not completely treat heteroscedasticity for valid inference since the sequence of violations is binary. They instead recommend estimating a logit model by assuming that the error term u_s has a logistic distribution and then testing the null hypothesis by a likelihood ratio test. Another possibility is a duration based approach developed by **?** and further by Berkowitz et al. (2011).

[23] Bailouts for American International Group, Inc. and Citi group Inc. were announced by the Federal Reserve Bank's Board of Governors. Though the case of General Motors Corporation is not technically a bailout, a bridge loan was given to the auto manufacturers by the U.S. government in 2008, which is referred to by most as a bailout.

be erroneous. Specifically, all the trades (quotes) with condition codes, A, C, D, G, L, N, O, R, X, Z, 8, 9 (4, 5, 7–9, 11, 13–17, 19, 20) are removed from the sample. The trades with a correction code which is greater than 2 are also removed from the sample.[24] Quotes are excluded if ask is equal or less than bid, or bid–ask spread is above 75% of mid-quote, or ask (bid) is more than double or less than half of the previous ask. Only trades reported from 9:30 AM to 4:00 PM are included. According to Lee and Ready (1991), trades happen five seconds earlier than reported time. Therefore, trade times are calculated as the reported time minus 5 seconds. Trades are deleted if the trade price is more than double or less than half of the previous trade. After filtering data, from the opening time 9:30 AM of the exchange, total trading volume, close trade and quote prices of every 5 minutes have been calculated based on the intraday transaction data. Close trade and quote prices in intervals are defined as the price of the last trade in intervals and corresponding quote price when trade happens. If there is no trade happening in the interval, the closest trade and quote prices of the previous interval will be used as the one for current interval. In addition, a portfolio return is constructed by the equal weight of the 30 company's returns. It is used for the proxy of the Dow Jones Industrial Average index. A detailed data description is provided by Table 2.

[TABLE 2 ABOUT HERE]

Table 3 presents the descriptive statistics for the intraday returns and the daily returns for the 30 companies and their equally weighted portfolio. In Panel A, the statistics for the intraday return are obtained by averaging the intraday statistics from 2000 to 2008. There are 78 five-minute-interval return observations in each day. The mean and median are very close to zero which is consistent with the previous market microstructure literature. The maximum return (0.92%) is observed in Alcoa Inc. and the minimum return (-0.94%) is observed in Intel Corporation. These statistics suggest the sudden increase and decrease of 1% in a five minute interval. The standard deviation is also biggest in Intel Corporation. The asymmetry is insignificant since skewness is small, close to zero. On the contrary, the kurtosis is significantly larger than that of the normal distribution: this describes the typical fat-tail of a financial time series.

In Panel B, the statistics for the daily return are obtained from 2,263 daily observations whilst those for Hewlett-Packard Company, AT&T, and Verizon Communication Inc. are from 1,678, 2,256 and 2,173 observations, respectively. The mean returns are distributed around zero. The means of American International Group, Inc. (-0.17%) and Citigroup (-0.08%) are relatively smaller than the others which is not a surprise given that they needed rescues by the government in 2008. The maximums (minimums) and standard deviations of those companies are also relatively much larger (smaller) than others. The asymmetry of the return distribution is generally not serious, whilst AIG (-6.49) and The Procter & Gamble Company (-5.17) are very negatively skewed. A fat-tail is also strongly observed for all companies. In particular, the kurtosis of AIG (158.5), Citigroup (47.82) and the Procter & Gamble Company (120.3) are significantly bigger than the others. The return of the portfolio shows the average property in both intraday and daily return except for the standard deviation. The standard deviation of the portfolio is least among the returns which reflects the diversification effect of the equal weighted portfolio. Overall, the returns present the typical nature of

[24] For the definitions of the codes, refer to the TAQ manual.

financial time series and some extreme statistics are observed especially for the bailout companies in the 2008 global crisis.

[TABLE 3 ABOUT HERE]

5 Evaluations

We evaluate the forecasting performance with a simulated sample, 30 individual stocks and their equal weighted portfolio. First, we evaluate the density forecasting of functional models such as FAR, AVE and LAST based on divergence criteria. This evaluation investigates which functional model is the best predictor of the density. Second, we evaluate the forecasting precision of the VaR models through a Monte Carlo simulation study in which we generate a sample based on a specification from empirical findings and we evaluate the forecasting precision by comparing a VaR forecast with a true VaR. Third, we evaluate the out-of-sample performance of VaR forecasting for alternative VaR models with 30 individual stocks and the portfolio. In the first part, the evaluations are performed for long and short positions with a window size of 250 business days, that is, we investigate the left-tail and the right-tail extreme events of the return distribution. Hence, this provides VaR forecasting evaluations for buying and selling activities. In the second part, we provide the effect of a longer window size on the performance of VaR forecasting, using for a 500 (business days) window size. These evaluations are also performed for long and short positions. Hence, it will better support our empirical evaluation results at the end.

5.1 The Density Forecasting of Alternative Functional Models

We first evaluate the density forecasting of alternative functional models; FAR, AVE and LAST—see (8) and (9). This evaluation demonstrates which functional model is the best predictor of the intraday density function using 30 individual stocks and portfolio. To this end we employ divergence criteria that measure the distance between the forecasted and the true density function; namely, the Hilbert norm (D_H), the uniform norm (D_U) and the generalised entropy (D_E). D_H and D_U are given by

$$D_H\left(\hat{f},f\right)=\frac{\int\left(\hat{f}(x)-f(x)\right)^2dx}{\int\hat{f}(x)^2dx+\int f(x)^2dx}\text{ and }D_U\left(\hat{f},f\right)=\frac{\sup_x\left|\hat{f}_t(x)-f_t(x)\right|}{\sup_x f_t(x)},\tag{46}$$

where $\hat{f}$ denotes the density forecast and f the true density. Following Ullah (1996), we define D_E by

$$D_E\left(\hat{f},f\right)=\int f(x)\,g\left(\frac{\hat{f}(x)}{f(x)}\right)dx,\tag{47}$$

where $g(y)=(\gamma-1)^{-1}(y^{\gamma}-1)$ with $\gamma>0$ and $\gamma\neq 1$. We follow Park and Qian (2007) and set $\gamma=1/2$. If g is the natural log, this becomes the Kullback–Liebler divergence measure. All three quantities are non-negative and give a zero value if $\hat{f}_t=f_t$ and so we may call them global errors. D_H is useful for evaluating the goodness-of-fit of the model, D_U is informative for comparing the closeness of the function shape, and D_E assesses the difference in information content between the forecast and the true density function.

For the 30 stocks and the portfolio, we apply the rolling one-step-ahead forecast of the intraday density function based on the 250 window size from 03/01/2001 to 31/12/2008 for three functional models. In Table 4, the mean and median divergence measures are calculated based on the 2,011 daily forecasts (except for Hewlett-Packard, AT&T and Verizon Communication Inc., which are calculated using 1,416, 2,004 and 1,885 forecasts, respectively). Panel A presents the average divergence values of 30 stocks and Panel B presents the divergence value of the portfolio. The FAR has the minimum values for the three divergence criteria in mean and median in Panels A and B. The LAST has smaller values than the AVE for the three divergence measures in mean and median. The results show that FAR outperforms other functional models in forecasting the intraday density function, demonstrating that FAR is the best predictor of the intraday density function in our sample.

[TABLE 4 ABOUT HERE]

5.2 The Precision of VaR Forecasting

In this section, we evaluate the precision of the VaR forecasting by comparing it with the true VaR via Monte Carlo simulation. To this end, we make the following assumptions on the data generating process:

Assumption 1. Intraday returns are independently and identically distributed.
Assumption 2. The distribution of intraday returns is determined by the first four moments such that it is a member of NIG family.
Assumption 3. The time-dependence of the first four moments of the intraday density function follows a stationary vector autoregressive process of lag order 1.
Assumption 4. The daily return is the sum of the intraday returns and follows a NIG distribution.

First, we construct the time-varying four moments of the intraday density function by a stationary VAR(1) process

$$\mathbf{m}_t - \mathbf{m} = \Phi\left(\mathbf{m}_{t-1} - \mathbf{m}\right) + \xi_t,\ t = 1, \ldots, T, \tag{48}$$

where $\mathbf{m}_t = (\mu_t, v_t, s_t, k_t)'$, $\mathbf{m} = (\mu, v, s, k)'$ and Φ is 4×4 autoregressive coefficient matrix satisfying $\left\|\Phi^k\right\| < 1$ for any $k \geq 1$. $\xi_t = (\xi_{1t}, \xi_{2t}, \xi_{3t}, \xi_{4t})'$ is independently and normally distributed with zero mean and covariance such that

$$\Sigma = \begin{bmatrix} \sigma_1^2 & 0 & 0 & 0 \\ 0 & \sigma_2^2 & 0 & 0 \\ 0 & 0 & \sigma_3^2 & 0 \\ 0 & 0 & 0 & \sigma_4^2 \end{bmatrix}.$$

In practice, we generate the sample by

$$\mathbf{m}_t = \mathbf{c} + \Phi \mathbf{m}_{t-1} + \xi_t, \tag{49}$$

where $\mathbf{c} = (\mathbf{I} - \Phi)\,\mathbf{m}$ and $\mathbf{I} - \Phi$ is invertible. Furthermore, the moments must satisfy the following conditions

(see the definition of NIG distribution):

$$v_t > 0 \text{ and } k_t > \frac{5}{3}s_t^2. \tag{50}$$

Let $\mathbf{m}_0 = \mathbf{0}$ and generate the first observation by

$$\mathbf{m}_1 = \mathbf{c} + \xi_1. \tag{51}$$

If the sample does not satisfy (50), then we regenerate it until it does. Other samples are sequentially generated by (49) in the same manner. We discard the first 100 observations to reduce the effect of the initial values.

Next, we calculate the four parameters $(\alpha_t, \beta_t, \gamma_t, \delta_t)'$, that determine the NIG distribution, using the four moments $(\mu_t, v_t, s_t, k_t)'$. Then we draw an intraday return from the NIG distribution and let $X_t := (r_{ti})_{i=1}^m$ and $X := (X_t)_{t=1}^{T+N}$ (See (3) for X_t and X). Under Assumptions 1 and 4, an NIG distribution for a daily return $(r_t = \sum_{i=1}^m r_{it})$ is determined by four parameters $(\alpha_t, \beta_t, m\gamma_t, m\delta_t)'$. Hence, we can calculate a true daily VaR for $(1-\alpha)\%$ from an inverse cumulative density function corresponding to α probability, $G_t^{-1}(\alpha)$.

We iterate the experiment 5,000 times and evaluate the precision of the VaR forecast for FAR and alternative models by comparing the VaR forecast and the true VaR. We employ Bias, Root Mean Square Error (RMSE), Mean Absolute Error (MAE), Mean Absolute Percentage Error (MAPE) and Percentage Mean Absolute Deviation (PMAD) as precision measures.[25]

The simulation uses a 250 window size $(T = 250)$ and one-step ahead forecast $(N = 1)$. We choose parameter values for $\mathbf{m}$, Φ and Σ by estimating (49) for the 30 stocks and the portfolio. Out of 31 estimations, we focus on three representative cases. We first consider the aggregate information of 30 stocks and select the portfolio that presents an equal weight composite of the 30 stocks, and obtain the following specifications for $\mathbf{m}$, Φ and Σ:

$$\mathbf{m} = \begin{bmatrix} -0.0003 \\ 0.0212 \\ 0.1666 \\ 9.4345 \end{bmatrix}, \Phi = \begin{bmatrix} -0.0495 & 0.0075 & 0.0000 & 0.0000 \\ -0.3850 & 0.5546 & 0.0019 & -0.0006 \\ -2.7238 & -0.3703 & -0.0623 & 0.0011 \\ 34.0111 & 12.2691 & -0.3776 & -0.0027 \end{bmatrix},$$

$$\Sigma = \begin{bmatrix} 0.0003 & 0 & 0 & 0 \\ 0 & 0.0023 & 0 & 0 \\ 0 & 0 & 3.2681 & 0 \\ 0 & 0 & 0 & 88.6670 \end{bmatrix}, \tag{52}$$

where $\mathbf{I} - \Phi$ is invertible and the maximum eigenvalue of Φ is 0.5337. Hence, this specification guarantees

[25] $Bias = \frac{\sum_{t=1}^N \left[\hat{G}_t^{-1}(\alpha) - G_t^{-1}(\alpha)\right]}{N}$, $RMSE = \sqrt{Bias^2 + Var\left(\hat{G}_t^{-1}(\alpha)\right)}$, $MAE = \frac{\sum_{t=1}^N \left|\hat{G}_t^{-1}(\alpha) - G_t^{-1}(\alpha)\right|}{N}$, $MAPE = \frac{\sum_{t=1}^N \left|\frac{\hat{G}_t^{-1}(\alpha) - G_t^{-1}(\alpha)}{G_t^{-1}(\alpha)}\right|}{N}$ and $PMAD = \frac{\sum_{t=1}^N \left|\hat{G}_t^{-1}(\alpha) - \hat{G}_t^{-1}(\alpha)\right|}{\sum_{t=1}^N \left|G_t^{-1}(\alpha)\right|}$.

the stationarity of the VAR(1) process.

Second, we consider a medium case for 30 stocks. Mean or median values of parameters are considered but the mean is severely affected by the outliers of the bailout companies such as AIG, Citi and GM. Hence, we select the median values of $\mathbf{m}$, Φ and Σ out of 30 estimations such that

$$\mathbf{m} = \begin{bmatrix} -0.0002 \\ 0.0581 \\ 0.1117 \\ 8.6073 \end{bmatrix}, \Phi = \begin{bmatrix} -0.0282 & 0.0027 & 0.0003 & -0.0001 \\ -0.2797 & 0.3434 & 0.0027 & -0.0016 \\ -2.1938 & 0.1015 & 0.0185 & -0.0015 \\ 5.2755 & 1.2691 & -0.0042 & 0.0286 \end{bmatrix},$$
$$\Sigma = \begin{bmatrix} 0.0007 & 0 & 0 & 0 \\ 0 & 0.0218 & 0 & 0 \\ 0 & 0 & 2.6799 & 0 \\ 0 & 0 & 0 & 75.7837 \end{bmatrix}, \tag{53}$$

where the maximum eigenvalue of Φ is 0.3355, guaranteeing the stationarity of the VAR(1) process.

Third, we consider the most persistent estimation case among 30 stocks. We search for a case presenting the maximum eigenvalue of Φ and finally select the case of Exxon Mobil Corporation where the values of $\mathbf{m}$, Φ and Σ are given by

$$\mathbf{m} = \begin{bmatrix} 0.0004 \\ 0.0420 \\ 0.1203 \\ 7.8954 \end{bmatrix}, \Phi = \begin{bmatrix} -0.1627 & 0.0260 & 0.0009 & 0.0000 \\ -0.4836 & 0.6382 & 0.0022 & -0.0009 \\ -4.3404 & -0.0316 & 0.0354 & -0.0011 \\ 4.8516 & 2.8674 & -0.1539 & 0.0427 \end{bmatrix},$$
$$\Sigma = \begin{bmatrix} 0.0005 & 0 & 0 & 0 \\ 0 & 0.0046 & 0 & 0 \\ 0 & 0 & 2.1028 & 0 \\ 0 & 0 & 0 & 51.0363 \end{bmatrix}, \tag{54}$$

where the maximum eigenvalue of Φ is 0.6165 which also guarantees the stationarity of the VAR(1) process.

Table 5 presents the evaluation results for alternative measures. Ten models are considered for each experiment and alternative positions (short and long) are taken into account for each model. For all experiments and measures, the results strongly conclude that FAR forecasts the true VaR most precisely. It provides the smallest values in the three experiments. RM, GARCH and CAViaR are less accurate than FAR while they are slightly more accurate than HS and FHS. The EVT models perform the worst in terms of precision. Their values are much greater than those of the other models. All models overestimate a true VaR and the biases are more or less symmetric for both positions. MAPE is quite sensitive to the extreme cases. The values of the long position are much greater than those of the short position. This implies that there are extreme negative realisations leading to the large forecasting error. Therefore, Monte Carlo simulation

study provides the evidence that FAR can precisely estimate and forecast a true VaR.

[TABLE 5 ABOUT HERE]

5.3 The Out-of-sample Performance of VaR Forecasting

Given the success with the forecasting precision of FAR compared with the true intraday density function and VaR, we evaluate the empirical VaR forecasting performance of FAR compared with alternative models. We apply various backtestings to the performance evaluation, since a true VaR is unobservable in practice. To this end, we apply the rolling-forecast of the daily VaR based on a 250 window size from 03/01/2001 to 31/12/2008 to the 30 stocks and the equal weighted portfolio. We also consider the evaluations of the alternative positions (long and short). The quantitative measures and the test statistics for the 30 stocks and the portfolio are calculated based on the 2,011 daily forecasts (except those for Hewlett-Packard, AT&T and Verizon Communication Inc., which are calculated using 1,416, 2,004 and 1,885 forecasts, respectively).

5.3.1 Long Position

We first evaluate the backtestings for the long position with 250 window size and the results are summarised in Table 6. Panel A reports the quantitative backtestings and Panel B the statistical backtesting for the 30 stocks and the portfolio. In Panel A, the rows of (A) empirical coverage probability, (C) MRCR and (D) predictive quantile loss present the average quantitative values of the 30 stocks. The (B) Basel penalty zone counts the number of stocks falling into the zone determined by the average violation number for the 99 percent VaR in the previous 250 business days. The (A) empirical coverage probability and the (B) Basel penalty zone generally evaluate the coverage ability of the model. Furthermore, the (C) MRCR and the (D) predictive quantile loss evaluate the economic cost for employing the model. The figures of (E)–(I) count the number of stocks that are rejected at the 5% significance level for each test.[26] Panel B presents the evaluation results for the portfolio. The row (A), (C) and (D) present the same figures as those in Panel A whilst the row (B) presents the penalty zone based on the average number of daily violations of the 99 percent VaR in the previous 250 business days. Furthermore, (E)–(I) present the test statistics and * , **, and *** denote the rejection of the null hypothesis for each test at, respectively, the 10%, 5% and 1% significance level. Those five test statistics employed in our evaluation are described in Section 3.2.

The quantitative backtestings of Panel A provide the following findings. For the (A) empirical coverage probability, FAR (1.07%) is closest to the 1% nominal probability. The FHS (1.10%) and the HS (1.19%) slightly underestimate the VaR. The GARCH models (1.60%–1.82%) considerably underestimate VaR whilst the EVT models (0.49%–0.55%) greatly overestimate VaR as expected. CAViaR (2.83%) shows the worst coverage ability by seriously underestimating the VaR. Overall, the hybrid non-parametric models demonstrate powerful coverage ability. For the (B) Basel penalty zone, the non-parametric models and the EVT models achieve the "Green" zone for all stocks. The non-parametric models obtain the "Green" zone due to their good coverage probability. The EVT models are however classified in the same zone due to their considerable overestimation of VaR. Since the Basel penalty zone is only concerned with underestimations

[26] We have obtained quantitatively similar results for all cases using the 10% significance level.

(violation) of VaR, the overestimation receives better scores by the Basel penalty rule given their prudential principles. The GARCH models fail to achieve the "Green" zone for some stocks because of their underestimations of VaR. GARCH achieves the "Green" zone for 20 stocks whilst the RM achieves it for seven stocks. The worst case, CAViaR, fails to obtain the "Green" zone for any of stocks. There are 3 and 27 stocks falling into the "Red" and the "Yellow" zone, respectively.

For the (C) MRCR, the GARCH models (39.59%–39.78%) demand the smallest capital requirement. FAR (40.80%) requires a similar level of capital whilst FHS (45.29%%) and HS (45.28%) demand much more capital. The capital requirement of adopting CAViaR (47.43%) is quite high although the highest level is found in the EVT models (58.09%–60.75%). For the (D) predictive quantile loss, FAR (6.76%) faces the smallest loss. The losses of GARCH (6.80%), FHS (6.87%) and RM (6.89%) are more or less similar to those of FAR. On the other hand, HS (7.74%), the EVT models (7.61%–8.21%) and CAViaR (8.25%) require banks to endure big losses.[27]

For the (E) unconditional coverage test, FHS is not rejected for any of stocks. HS and FAR are rejected for 3 and 7 stocks, respectively. Other models are rejected for over 20 stocks. These results imply that the hybrid models utilising a non-parametric distribution have statistically significant coverage ability whilst others have very poor ability. For the (F) independence test, the coverage ability of HS and CAViaR is statistically more dependent on the underlying states than other models. HS and CAViaR are rejected for 9 and 10 stocks. Hence, the violations of VaR forecasts are considerably time dependent on the underlying state. Regulators would therefore prefer the models allowing for a time-varying distribution which would prevent the violations from being abnormally clustered. The FAR would be the best candidate among such models since it is rejected for only 2 stocks. RM is rejected for 3 stocks, GARCH, FHS, and unfiltered EVT models are rejected for 4 stocks and the FEVT models are rejected for 6 stocks.

From the above two tests, the hybrid non-parametric models achieve statistically powerful coverage ability that is independent of the underlying state.[28] Other models fail to pass the two tests jointly. These findings are further supported by the (G) conditional coverage test. The hybrid non-parametric models are rejected for 4 or fewer stocks whilst the others are rejected for 9 or more stocks (ranging from HS: 9 stocks to CAViaR: 30 stocks). There are alternative types of conditional tests such as the (H) dynamic quantile test 1 and the (I) dynamic quantile test 2. The (H) dynamic quantile test 1 includes the past violations in the past information set and the (I) dynamic quantile test 2 additionally includes the past VaR forecast in the past information set. In the (H) dynamic quantile test 1, the hybrid non-parametric models and the FEVT models are rejected for a relatively small number of stocks whilst the others are rejected for over the half of the stocks. Especially, GARCH (22 stocks), HS (24 stocks), RM (28 stocks) and CAViaR (30 stocks) are rejected for over 70% of the 30 stocks. These results resemble the (I) dynamic quantile test 2. Overall, the

[27] For example, suppose that a bank manages $1 bn. and operates an internal VaR model. Then the CFO will report the economic cost using the MRCR and the predictive quantile loss every day. If the bank operates the EVT models, they should allocate $580.9 mm.–$607.5 mm. against the maximum loss for 10-day or endure $76.1 mm.–$82.1 mm. of loss every day. Then the CFO will decide to replace the EVT models with FAR, since the EVT models are too expensive to operate. The FAR reduces the capital requirement to $408 mm. and the loss to $67.6 mm. Therefore, FAR cuts down the capital requirement by $172.9 mm.–$199.5 mm. and the loss by $35.3 mm.–$41.3 mm.

[28] This means that the models provide the closest coverage probability to the 1% nominal probability regardless of the underlying state such as business cycle or volatility clustering.

hybrid non-parametric models statistically perform better than other models.

Panel B presents the evaluation results for the portfolio. The (A) empirical coverage probabilities of non-parametric models (0.80%–1.24%) are relatively closest to the 1% nominal probability. This finding confirms that the models utilising the non-parametric distribution have stronger coverage ability than those of the models utilising the parametric distribution. The GARCH models (1.79%–1.84%) and CAViaR (2.09%) considerably underestimate VaR whilst the EVT models (0.45%–0.55%) greatly overestimate it. These lead to that the adoption of the GARCH models and the CAViaR will be classified as the "Yellow" penalty zone. For the (C) MRCR, the GARCH models (24.64%–25%) demand the smallest capital requirements and the non-parametric models (25.89%–28.84%) and the CAViaR (26.66%) require slightly more capital. Banks would be required to pay an expensive cost to adopt the EVT models (35.47%–35.56%). The (D) predictive quantile loss of FAR and the GARCH models (4.38–4.44%) are smallest whilst the EVT models (4.98%–5.18%) face the biggest loss. Those of the FHS, HS and CAViaR (4.48%–4.61%) are in the intermediate level. These results confirm that the non-parametric models are superior to the others in terms of their coverage ability and economic cost. The GARCH models and CAViaR reduce the economic cost well whilst they fail to increase the coverage ability. The EVT models present very poor performance in both coverage ability and economic cost analysis. Moreover, the statistical evaluations from (E)–(I) show that the non-parametric models are not rejected for all the tests whilst CAViaR is rejected for all of them. The GARCH models also fail for all the tests except for the (F) independence test. The FEVT models are not rejected for all the conditional tests at the weak significance level whilst the unfiltered EVT models are rejected for all the tests except for the (F) independence test. These results further confirm that the non-parametric models have statistically significant and robust coverage ability that is independent of the underlying state.

[TABLE 6 ABOUT HERE]

5.3.2 Short Position

The results for the short position, summarised in Table 7, are generally consistent with those for the long position. The results for the quantitative backtestings in Panel A provide the following. For the (A) empirical coverage probability, the hybrid non-parametric models (0.96%–1.03%) are closest to the 1% nominal probability. FAR (0.96%) slightly overestimates the VaR whilst FHS (1.03%) slightly underestimates it. HS (1.20%) underestimates even more the VaR. The GARCH models (1.46%–1.66%) underestimate VaR whilst the EVT models (0.48%–0.56%) considerably overestimate it. CAViaR (2.31%) seriously underestimates the VaR. For the (B) Basel penalty zone, FHS achieves the "Green" zone for all stocks. FAR and HS achieve the "Green" zone for 29 stocks and the "Yellow" zone for 1 stock. The EVT models obtain "Green" zone for 30 stocks, since the models much overestimate VaR. The GARCH models fail to achieve the "Green" zone for some stocks because of their severe underestimations of VaR. GARCH achieves the "Green" zone for 21 stocks whilst RM achieves it for 14 stocks. The worst case is CAViaR, which obtains the "Green" zone for only one stock and the "Yellow" zone for 28 stocks. For the (C) MRCR, the GARCH models require the smallest capital (38.64%–39.04%) and the hybrid non-parametric models require a bit more capital

(41.20%–42.88%). The EVT models (52.53%–56.51%) are the most expensive models for banks to adopt as their internal VaR models. The capital requirements for HS (45.13%) and CAViaR (44.13%) are somewhat higher. For the (D) predictive quantile loss, FAR (5.93%) has the smallest loss followed by GARCH (5.97%), FHS (6.05%) and RM (6.09%). On the other hand, HS (7.06%), the EVT models (6.60%–7.44%), and CAViaR (7.10%) require banks to endure big losses. The results of the quantitative backtestings show that the hybrid non-parametric models obtain a powerful coverage ability at small economic cost. The GARCH models and HS fail to obtain a strong coverage ability and a small economic cost jointly.

For the (E) unconditional coverage test, FHS is not rejected for any of the stocks. HS and FAR are rejected for three and five stocks, respectively. Other models are rejected for over 20 stocks except for GARCH, rejected for 14 stocks. For the (F) independence test, overall, the coverage ability of VaR forecasts are independent of the underlying state. The hybrid non-parametric models and the GARCH models are rejected for one company at most. Other models are rejected for slightly more stocks but less than four stocks. These results imply that the non-parametric models have statistically powerful coverage ability that is more or less statistically robust to the underlying states but other models fail to pass the two tests jointly. These findings are further confirmed in the (G) conditional coverage test. FHS is rejected for the none of 30 stocks and FAR is rejected for four stocks. HS is rejected for seven stocks, which is relatively small compared with other models (ranging from GARCH: 12 stocks to CAViaR: 30 stocks). Moreover, for the (H) dynamic quantile test 1, the hybrid non-parametric models and the FEVT models are rejected for five or fewer stocks (ranging from FHS: one company, to FGEV: five stocks) whilst others are rejected for more stocks. Especially, HS (16 stocks), RM (20 stocks) and CAViaR (30 stocks) are rejected over the half of the 30 stocks. These results are consistently observed in the (I) dynamic quantile test 2. Therefore, the hybrid non-parametric models provide statistically the most powerful coverage ability, which is independent of the underlying state.

In Panel B, the (A) empirical coverage probability of FAR (0.99%) is closest to the 1% nominal probability. GARCH (1.09%) and FHS (1.14%) somewhat underestimate VaR. HS (1.79%) and CAViaR (1.59%) much underestimate the VaR whilst the FEVT models (0.25%–0.35%) greatly overestimate it. These lead to considering that adoption of the HS and CAViaR models will be classified in the "Yellow" zone. The "Green" zone of the EVT models is completely due to the considerable overestimation of VaR, which is misleading. For the (C) MRCR, the GARCH models (23.98%–24.42%) require the smallest capital. The hybrid non-parametric models (25.19%–27.92%), CAViaR (24.77%), and the FEVT models (28.38%–28.62%) require slightly more capital. The unfiltered EVT models (33.44%–33.50%) have the most expensive cost of adoption. The (D) predictive quantile loss of the GARCH models (3.31%–3.34%) are the smallest whilst the losses are largest for HS and the unfiltered EVT models (4.43%–4.49%). Those of the hybrid non-parametric models, the FEVT models and CAViaR (3.35%–3.65%), are slightly more than those of the GARCH models. These results confirm that the hybrid non-parametric models are superior to the others in terms of coverage ability and reasonable economic cost. The GARCH models, the FEVT models and CAViaR are successful at reducing the economic cost but fail to achieve adequate coverage ability. Furthermore, the unfiltered EVT models are undesirable from the results of both evaluations. Moreover, the statistical evaluations from (E) to (I) show that the hybrid non-parametric models are successful for all the

tests. HS, the EVT models and CAViaR are strongly rejected for two tests at least. The GARCH models fail for only (I) dynamic quantile test 2. Hence, the statistical backtestings further confirm that the coverage ability of the hybrid non-parametric models are accurate and state-independent.

[TABLE 7 ABOUT HERE]

Stylised Facts

We can draw the following conclusions from the quantitative and the statistical backtestings. First, the hybrid non-parametric models, FAR and FHS, perform better in modelling the tail behaviour than purely non-parametric and parametric models do. Second, the GARCH models with normal/Student's t-distributions considerably underestimate VaR whilst the EVT models much overestimate it. The normal distribution generally has a thinner tail than the distribution of financial time series, which limits its ability in capturing extreme risk. Furthermore, both the normal distribution and Student's t-distribution are symmetric so that they cannot detect the skewness which is frequently observed in financial time series (Theodossiou, 1998). The GARCH models building on the assumption of these two distributions therefore tend to underestimate the tail risk. Furthermore, the GARCH models perform worse for a long position than for a short position, since the models cannot detect the negative skewness frequently observed in individual stocks. On the other hand, the EVT models estimate VaR by capturing the extreme of the extreme values which usually tends to be greater than the true extreme values. Hence, the EVT models can cover the loss very well in the period of a financial crash whilst it often much overestimates the potential loss in a normal period. Furthermore, there is always the trade-off between the number of observations and the degree of extremeness in applying EVT models.[29] Third, the dynamic models considering the time-varying distributions perform better than the static models such as HS and the unfiltered EVT models. Especially, the coverage ability of the dynamic models is much more independent of the underlying state than those of the static models. Furthermore, HS and the unfiltered EVT models are improved by the GARCH filtering of time-series. Finally, the above conclusions are consistently observed in both long and short positions. Hence, our proposed hybrid non-parametric models are quite robust in forecasting VaR.

5.4 The Effect of Window Size

We repeat the previous analysis for the window size of 500, as a robustness check. The quantitative measures and test statistics for the 30 stocks and the portfolio are calculated based on the 1,763 daily forecasts (except for Hewlett-Packard, AT&T and Verizon Communication Inc., which are calculated using 1,178, 1,756 and 1,637 forecasts, respectively). The evaluation results are presented in Tables 8 and 9. Table 8 presents the results for the long position and Table 9 presents the results for the short position. Overall, the results are consistent with those for the shorter window size.

[29] The EVT estimates are affected by a small number of observation. If we set a wide block size for extreme observations, the number of extreme observations is not enough to estimate the parameters accurately. If we obtain enough observations by setting a narrow block size for extreme observations, the degree of extremeness is weaker.

5.4.1 Long Position

Table 8 reports the results for the long position. The quantitative backtesting results, summarised in Panel A, are strongly consistent with those of the shorter window size. The hybrid non-parametric models outperform other models in terms of the coverage ability and the economic cost. The GARCH models reduce the economic cost but fail to improve the coverage ability. The EVT models fail both at improving the coverage ability and at reducing the economic cost. The notable effects of the longer window size are observed in HS and the EVT models. First, the EVT models greatly overestimate VaR compared with the shorter window size. This is driven by the degree of extremeness becoming stronger as the window size increases. On the other hand, HS considerably underestimates VaR with the longer window size. As we use the longer history to estimate a non-parametric distribution, the weight on recent regime changes such as volatility clustering is reduced. Hence, the underestimation would be frequently observed during the "Recession".[30] Second, these inherent limitations of the EVT models and HS also make for an increase in the economic cost. Those of the EVT models most significantly increase and the magnitudes are much larger than those of the shorter window size. The performance of CAViaR improves a bit and the hybrid non-parametric models performs similarly with the shorter window size. The statistical evaluations of Panel A are also almost consistent with those of the shorter window size. The hybrid non-parametric models provide statistically desirable performance for banks and regulators to adopt the models as their internal VaR models. Generally, the rejection frequency decreases for all the tests except for HS. However, there are no significant changes in that the hybrid non-parametric models still outperform other models.

Panel B presents the evaluations for the portfolio. Based on the quantitative evaluations, the hybrid non-parametric models and CAViaR present good coverage ability and small economic cost. The GARCH models achieve a small economic cost whilst they still suffer from poor coverage ability by reason of their considerable underestimation of VaR. The EVT models obtain neither powerful coverage ability nor small economic cost by reason of their huge overestimation of VaR. Hence, the results of the quantitative evaluations are more or less consistent with those for the shorter window size. Moreover, the statistical evaluations show that the hybrid non-parametric models and CAViaR are successful for all tests. The FEVT models are strongly rejected for the (E) unconditional coverage test and the (G) conditional coverage test. Other models are rejected for all tests except for the (F) independence test. The results of the statistical evaluations are consistent with those of the quantitative evaluations. Furthermore, the results are more or less consistent with those for the shorter window size.

In sum, the quantitative and the statistical evaluations confirm that the hybrid non-parametric models provide powerful coverage ability which is independent of the underlying state. Generally, the performance of the models somewhat improves as the window size increases whilst the HS and the EVT models still suffer from their inherent limitations.

[TABLE 8 ABOUT HERE]

[30] For example, the recent regime indicates the "Recession" but HS includes the past information of the "Boom" and gives the same weight to all the historical events. Hence, HS is likely to underestimate VaR for the "Recession".

5.4.2 Short Position

The results for the short position are summarised in Table 9. The quantitative backtestings, summarised in Panel A, are overall consistent with those of the long position. Furthermore, those are strongly consistent with the shorter window size. The hybrid non-parametric models outperform other models in terms of coverage ability and economic cost. The GARCH models are successful at reducing the economic cost but fail to improve the coverage ability. The EVT models fail to improve the coverage ability and yield a big economic cost by the reason of their considerable overestimation of VaR (see Section 5.2.2 for the detailed explanation of the longer window size effect). The statistical evaluations also present that the hybrid non-parametric models provide powerful coverage ability that is independent of the underlying state, which is consistently observed in the three conditional tests. The hybrid non-parametric models provide the statistically desirable performance for banks and the regulators in adopting the models as their internal VaR models.

The evaluations for the portfolio, summarised in Panel B, presents that the hybrid non-parametric models, GARCH and CAViaR have good coverage ability and small economic cost. The FEVT models obtain more or less a small economic cost but fail to obtain a good coverage ability. The unfiltered EVT models get neither a powerful coverage ability nor a small economic cost. Moreover, the statistical evaluations show that the hybrid non-parametric models, the GARCH models and CAViaR are successful for all the tests. The EVT models are strongly rejected by two or more tests and HS is strongly rejected by four tests. The overall results of the quantitative and statistical evaluations are more or less consistent with those for the shorter window size.

Overall, the quantitative and the statistical evaluations confirm that the hybrid non-parametric models provide a powerful coverage ability that is independent of the underlying state. Other models such as the GARCH models and CAViaR provide an improved performance especially for the portfolio but they still provide inferior performance for the 30 stocks on average. The performance of the models slightly improves as the window size increases for the short position. However, the finding that the hybrid non-parametric models outperform other models is consistently observed.

[TABLE 9 ABOUT HERE]

Stylised Facts

We can draw the following conclusions for the case with the larger window size. First, the hybrid non-parametric models, FAR and FHS, perform better in modelling the tail behaviour than purely non-parametric and parametric models. This finding is consistently observed for both the long and the short position. Second, the performance generally improves as the window size increases for the long and the short position except for HS and the EVT models. HS more considerably underestimates VaR with the longer window size. If we use the longer history to estimate a non-parametric distribution, the weight on the recent regime such as volatility clustering is reduced. Hence, such underestimation would be frequently observed during the "Recession" by construction. On the other hand, the EVT models more greatly overestimate VaR with the longer window size, which is driven by the degree of extremeness becoming stronger as the window size

increases. Therefore, we confirm that the hybrid non-parametric models outperform other models for the case of the longer window size.

6 Conclusion

This study introduces a novel econometric technique in order to improve VaR analysis in terms of coverage ability and economic cost. Our proposed model complements pure non-parametric and parametric approaches by applying FAR to estimating and forecasting the intraday return density function.

We perform extensive evaluations on the performance of FAR modelling for the density function. First, we demonstrates that FAR is the best predictor of the intraday density function compared with alternative functional models such as AVE and LAST, with 30 stocks of DJIA and an equal-weighted portfolio. Second, a Monte Carlo simulation study provides the evidences that FAR can more precisely estimate and forecast a true VaR than other existing models: HS, FHS, GARCH, RM, EVT models, FEVT models, and CAViaR. Third, a wide range of quantitative and statistical backtestings are applied to the alternative VaR models and the results point strongly favour the hybrid non-parametric approaches which includes our FAR approach and FHS.

Overall, our extensive horse-racing of VaR models contributes to the understanding of VaR analysis in the following ways. First, FAR describes the dynamics of the intraday return density well. It is a generalisation of all autoregressive specifications. Hence, it considerably removes the uncertainty of existing parametric autoregressive models. Second, a non-parametric density function associated with a dynamic structure is superior to a parametric density function in estimating and forecasting risk. This is driven by the strong coverage ability of the non-parametric approach being independent of the underlying state in time-varying modelling. Third, a hybrid scheme reduces the economic cost and improves the coverage ability. The coverage ability is mainly determined by the robust estimation of the VaR and the economic cost by the time-independence from the underlying state. The hybrid scheme absorbs the robustness from the non-parametric approach and the time-independence from the dynamic modelling. Fourth, intraday information is helpful in forecasting the daily risk. The intraday data possesses important information which is relevant to market participants in forming their future expectations. Understanding the intraday return distribution is key to access the accumulated results of the daily return distribution. Hence, an accurate and well organised modelling of the intraday return distribution helps to estimate and forecast the daily risk.

There are several unanswered questions raised by this study which cannot be explained further due to the limited space and the focus of our study. First, we will show that individual VaR measures perform differently under normal and extreme market condition. It would be possible to develop a systematic approach of switching the use of measures conditional on the market condition. This switching will provide the jump necessary for the sudden change of a company's risk profile when they are approaching bankruptcy. This will require a Bayesian learning approach or Markov switching modelling of FAR. Second, VaR models capture risk spillovers indirectly to the extent that institutions are exposed to common risk factors, but they do not provide explicit information about the co-dependence of the risk. Given the increased interest in systemic risk, it would be desirable to extend our analysis into the co-dependence of risk between indi-

vidual institutions and the financial system. We can forecast CoVaR (Adrian and Brunnermeier, 2010) by applying FAR to the conditional quantile function of the financial system conditional on the individual institution. Hence, this will contribute to forecasting the systemic risk. Furthermore, Copula will be required for analysing the co-dependence of the risk with FAR. Finally, VaR analysis will call for a robust statistical test, since the existing tests usually suffer from low power.

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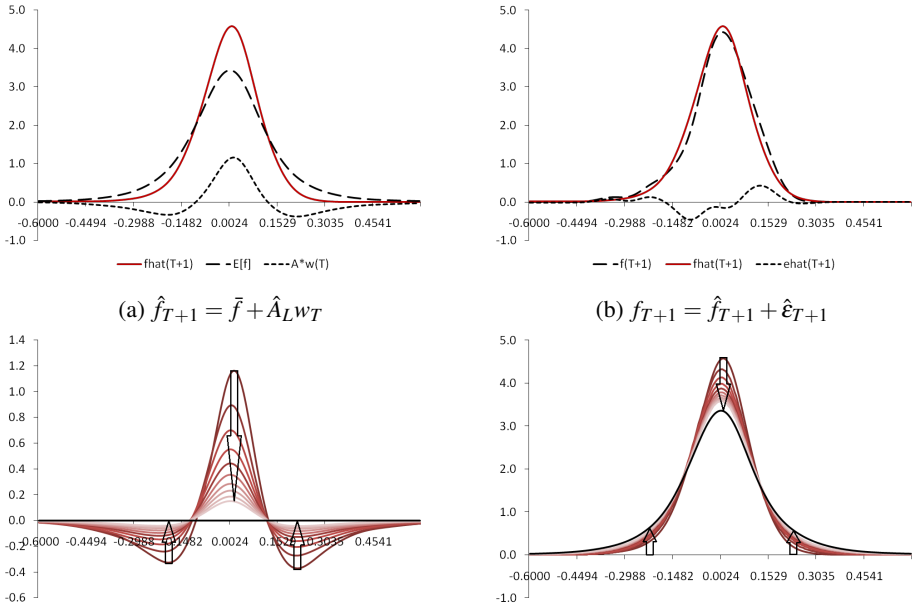
Figure 1: Functional Autoregressive Modelling of Non-parametric Intraday Density Function

The figures present the intraday return density forecast of the equal weighted DJIA portfolio on the 24th May 2001. 250 past densities are used for the estimation of FAR. Panel (a) presents that the density forecast $\left(\hat{f}_{T+1}\right)$ consists of the mean of the density $\left(\bar{f}\right)$ and the correction$\left(\hat{A}_L w_T\right)$ in (17). Panel (b) shows the forecasting error $\left(\hat{\varepsilon}_{T+1}\right)$. Panel (c) shows that the fluctuation $\left(w_T\right)$ gradually disappears as time period increases and is completely gone at the end. Panel (d) shows the mean reversion of the density function such that the density forecast converges to the expectation of the density function as the time period increases.



(a) $\hat{f}_{T+1} = \bar{f} + \hat{A}_L w_T$

(b) $f_{T+1} = \hat{f}_{T+1} + \hat{\varepsilon}_{T+1}$

(c) Correction of Fluctuation

(d) Mean Reversion of Density Function

Table 1: Summary of VaR Models

There are ten VaR models which compete each other in our study. They are categorised by modelling style: (i) parametric approach, (ii) non-parametric approach, (iii) empirical quantile approach and (iv) hybrid non-parametric approach. Our suggested VaR model, FAR, is included in the hybrid non-parametric approach.

Model	Name	Category
(1) FAR	Functional Autoregressive Modelling of Densities	Hybrid approach
(2) FHS	Filtered Historical Simulation	Hybrid approach
(3) HS	Historical Simulation	Non-parametric approach
(4) RM	RiskMetrics	Parametric approach
(5) GARCH	GARCH with t-distribution	Parametric approach
(6) GEV	Generalised Extreme Value Distribution	Parametric approach
(7) GPD	Generalised Pareto Distribution	Parametric approach
(8) FGEV	Filtered Generalised Extreme Value Distribution	Parametric approach
(9) FGPD	Filtered Pareto Distribution	Parametric approach
(10) CAViaR	Conditional Autoregressive Quantile	Empirical quantile approach

Table 2: Data Description for 30 Components of DJIA and Portfolio

30 companies of DJIA are collected from Trade and Quote (TAQ) and the portfolio is constructed by equally weighting their returns.

	Company	From	To	Open	Close
(1)	Alcoa Inc.	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(2)	American International Group, Inc.	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(3)	American Express Company	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(4)	The Boeing Company	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(5)	Citi group Inc.	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(6)	Caterpillar Inc.	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(7)	E.L. Du Pont de Nemours & Company	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(8)	Walter Disney Company	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(9)	General Electric Company	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(10)	General Motors	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(11)	Home Depot, Inc.	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(12)	Honeywell International Inc.	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(13)	Hewlett-Packard Company	06-May-02	31-Dec-08	9:30 AM	4:00 PM
(14)	International Business Machine Corp.	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(15)	Intel Corporation	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(16)	Johnson & Johnson	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(17)	J.P. Morgan Chase & Co.	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(18)	The Coca-Cola Company	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(19)	McDonald's Corporation	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(20)	3M Company	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(21)	Altria Group Inc.	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(22)	Merck & Co, Inc.	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(23)	Microsoft Corporation	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(24)	Pfizer, Inc.	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(25)	The Procter & Gamble Company	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(26)	AT&T	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(27)	United Technology Corporation	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(28)	Verizon Communication Inc.	03-Jul-00	31-Dec-08	9:30 AM	4:00 PM
(29)	Wal-Mart Stores, Inc.	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
(30)	Exxon Mobil Corporation	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM
	Portfolio	03-Jan-00	31-Dec-08	9:30 AM	4:00 PM

Table 3: Descriptive Statistics of Intraday Return and Daily Return for 30 Components of DJIA and Portfolio

Panel A is the average statistics of the intraday statistics. We assume that the intraday return is identically and independently distributed in the given trading day whilst the intraday return distribution is varies over time. We thus first calculate the intraday statistics for each day then we average the intraday statistics. The statistics for the daily returns are calculated using 2,263 trading days, except Hewlett-Packard Company, AT&T and Verizon Communication Inc. are calculated using 1678, 2256 and 2137 trading days, respectively. The statistics present the number of observations (T, N), mean (Mean), median (Med.), maximum (Max.), minimum (Min.), standard deviation (Stdev.), skewness (SK) and kurtosis (K) separately for the intraday return and the daily return.

	Panel A: Intraday Return								Panel B: Daily Return							
Company	T	Mean	Med.	Max.	Min.	Stdev.	SK	K	N	Mean	Med.	Max.	Min.	Stdev.	SK	K
(1)	78	0.00	0.00	0.92	-0.88	0.26	0.14	9.13	2263	-0.06	-0.03	20.88	-18.16	2.71	-0.06	10.77
(2)	78	0.00	0.00	0.83	-0.86	0.24	0.11	8.66	2263	-0.17	-0.06	53.90	-86.22	3.99	-6.49	158.5
(3)	78	0.00	0.00	0.77	-0.76	0.23	0.12	8.59	2263	-0.04	-0.02	16.49	-17.31	2.41	-0.37	9.26
(4)	78	0.00	0.00	0.72	-0.71	0.21	0.12	8.95	2263	0.00	0.03	14.53	-19.58	2.10	-0.33	9.37
(5)	78	0.00	0.00	0.86	-0.80	0.24	0.18	8.89	2263	-0.08	-0.05	47.00	-32.08	2.97	0.58	47.82
(6)	78	0.00	0.00	0.74	-0.73	0.22	0.12	8.82	2263	0.03	0.04	11.66	-15.33	2.08	-0.21	7.28
(7)	78	0.00	0.00	0.66	-0.66	0.20	0.07	8.16	2263	-0.04	-0.06	9.82	-12.27	1.88	-0.22	8.96
(8)	78	0.00	0.00	0.74	-0.78	0.22	-0.03	8.67	2263	-0.01	0.00	14.31	-20.12	2.22	-0.05	11.87
(9)	78	0.00	0.00	0.66	-0.65	0.19	0.12	8.67	2263	-0.05	-0.06	12.67	-13.88	2.03	-0.22	9.47
(10)	78	0.00	0.00	0.58	-0.57	0.18	0.12	7.89	2263	0.03	0.09	16.79	-15.92	1.79	0.15	14.63
(11)	78	0.00	0.00	0.79	-0.78	0.23	0.06	8.94	2263	-0.05	-0.05	12.79	-33.52	2.40	-1.11	22.44
(12)	78	0.00	0.00	0.84	-0.81	0.24	0.12	9.09	2263	-0.03	-0.01	19.42	-20.55	2.38	-0.48	13.30
(13)	78	0.00	0.00	0.79	-0.81	0.23	-0.05	10.1	1678	0.04	0.02	13.74	-16.21	2.25	0.13	9.91
(14)	78	0.00	0.00	0.65	-0.61	0.19	0.14	9.10	2263	-0.01	0.00	12.26	-17.01	1.94	-0.05	10.35
(15)	78	0.00	0.00	0.96	-0.94	0.28	0.13	9.66	2263	-0.05	0.00	18.20	-24.81	2.95	-0.49	9.49
(16)	78	0.00	0.00	0.49	-0.49	0.14	0.06	8.44	2263	0.01	0.00	8.94	-17.71	1.43	-0.84	18.52
(17)	78	0.00	0.00	0.84	-0.83	0.25	0.07	8.97	2263	0.00	0.00	16.95	-20.14	2.64	0.03	11.91
(18)	78	0.00	0.00	0.51	-0.54	0.16	-0.05	7.92	2263	-0.01	-0.02	9.94	-10.51	1.54	0.08	9.12
(19)	78	0.00	0.00	0.66	-0.67	0.20	0.05	8.41	2263	0.02	0.03	8.88	-13.20	1.78	-0.19	7.61
(20)	78	0.00	0.00	0.57	-0.54	0.17	0.13	7.84	2263	0.01	-0.03	9.99	-9.42	1.59	0.19	7.99
(21)	78	0.00	0.00	0.64	-0.59	0.18	0.18	8.57	2263	0.04	0.06	15.12	-14.37	1.88	-0.02	12.94
(22)	78	0.00	0.00	0.66	-0.69	0.19	0.04	9.10	2263	-0.04	0.00	11.84	-31.14	2.05	-1.83	31.32
(23)	78	0.00	0.00	0.72	-0.68	0.21	0.12	8.51	2263	-0.05	-0.04	17.77	-16.94	2.25	-0.08	11.07
(24)	78	0.00	0.00	0.65	-0.65	0.19	0.09	8.64	2263	-0.03	-0.04	9.48	-11.46	1.81	-0.24	7.34
(25)	78	0.00	0.00	0.48	-0.51	0.15	-0.05	7.61	2263	0.01	0.02	9.83	-37.44	1.66	-5.19	120.3
(26)	78	0.00	0.00	0.78	-0.79	0.23	0.04	8.90	2256	-0.05	-0.05	22.18	-21.00	2.44	0.28	14.90
(27)	78	0.00	0.00	0.65	-0.62	0.19	0.12	8.25	2263	0.03	0.03	12.31	-15.75	1.91	-0.16	9.55
(28)	78	0.00	0.00	0.66	-0.66	0.20	0.03	8.54	2137	-0.02	-0.02	13.28	-12.22	1.88	0.17	8.28
(29)	78	0.00	0.00	0.63	-0.62	0.19	0.11	8.43	2263	-0.01	-0.03	10.20	-8.31	1.79	0.26	6.88
(30)	78	0.00	0.00	0.58	-0.57	0.18	0.12	7.89	2263	0.03	0.09	16.79	-15.92	1.79	0.15	14.63
Portfolio	78	0.00	0.00	0.39	-0.39	0.12	0.17	9.43	2263	-0.02	0.02	10.95	-16.45	1.39	-0.75	18.66

Table 4: Forecasting Performance of Alternative FAR models by Three Divergence Criteria

D_H, D_U and D_E denote the Hilbert norm, uniform norm and entropy, respectively. Figures are evaluated by the mean and the median value between one-step-ahead density forecast and the true density for each of the three models.

	Mean			Median		
Models	D_H	D_U	D_E	D_H	D_U	D_E
Panel A: Average of 30 Components of Dow Jones Industry						
FAR	0.0284	0.2664	0.0406	0.0160	0.2044	0.0304
AVE	0.0473	0.3812	0.0601	0.0235	0.2493	0.0393
LAST	0.0353	0.2872	0.0557	0.0218	0.2335	0.0431
Panel B: Portfolio						
FAR	0.0276	0.2634	0.0465	0.0150	0.1992	0.0344
AVE	0.0568	0.4438	0.0916	0.0246	0.2534	0.0552
LAST	0.0320	0.2706	0.0600	0.0203	0.2245	0.0470

Table 5: Monte Carlo Simulation Study for Evaluating The Precision of VaR Forecast

Intraday returns are generated by the NIG distribution using four moments that follow VAR(1) process (see Assumptions 1–4 for details). We generate 78 observations for each day and 251 days for conducting one-step-ahead forecast for both long- and short-position conditional on the past 250 days. The simulation study replicates this forecasting 5,000 times with ten models and evaluates the forecasting precision based on Bias, RMSE, MAE, MAPE and PMAD. Three representative experiments are considered out of 31 estimations. Experiment I considers the aggregate information of 30 stocks and select the portfolio that presents an equal weight composite of 30 stocks. Experiment II considers a medium case for 30 stocks. Mean or median values of parameters are considered but the mean is severely affected by outliers of bailout companies such as AIG, Citi and GM. Experiment III consider the most persistent estimation case among 30 stocks, Exxon Mobil Corporation.

	Bias		RMSE		MAE		MAPE		PMAD	
Models	Long	Short	Long	Short	Long	Short	Long	Short	Long	Short
Experiment I										
FAR	-0.63	-0.63	1.04	0.98	1.79	1.81	1.80	1.74	0.39	0.40
FHS	-1.62	-1.17	1.85	1.41	2.23	2.00	2.17	2.01	0.49	0.44
HS	-1.65	-1.21	1.83	1.39	2.23	2.01	2.20	2.03	0.49	0.44
RM	-0.96	-0.95	1.26	1.26	1.92	1.94	1.87	1.93	0.42	0.42
GARCH	-1.03	-1.02	1.15	1.15	1.89	1.90	1.92	1.95	0.41	0.42
GEV	-2.96	-3.04	3.20	3.28	3.17	3.25	2.87	2.87	0.69	0.71
GPD	-2.87	-2.97	3.12	3.22	3.10	3.20	2.81	2.83	0.68	0.70
FGEV	-2.90	-2.97	3.18	3.25	3.14	3.19	2.80	2.83	0.69	0.70
FGPD	-2.82	-2.91	3.12	3.20	3.08	3.15	2.75	2.80	0.67	0.69
CAViaR	-1.10	-1.13	1.97	2.02	2.25	2.30	2.15	2.00	0.49	0.50
Experiment II										
FAR	-0.57	-0.52	0.90	0.86	2.83	2.85	2.03	1.87	0.38	0.38
FHS	-2.66	-1.89	3.01	2.26	3.64	3.32	2.54	2.27	0.48	0.45
HS	-2.68	-1.92	2.98	2.22	3.63	3.31	2.54	2.28	0.48	0.44
RM	-1.51	-1.50	2.01	2.01	3.17	3.20	2.17	2.13	0.42	0.43
GARCH	-1.60	-1.60	1.78	1.78	3.09	3.12	2.25	2.15	0.41	0.42
GEV	-4.90	-4.94	5.27	5.32	5.24	5.31	3.20	3.21	0.70	0.71
GPD	-4.74	-4.78	5.16	5.19	5.12	5.19	3.14	3.17	0.68	0.70
FGEV	-4.85	-4.90	5.27	5.33	5.22	5.29	3.19	3.18	0.69	0.71
FGPD	-4.70	-4.75	5.15	5.22	5.11	5.18	3.15	3.14	0.68	0.69
CAViaR	-1.73	-1.83	3.29	3.37	3.76	3.80	2.24	2.42	0.50	0.51
Experiment III										
FAR	-0.72	-0.65	1.10	1.01	2.11	2.20	7.71	1.70	0.37	0.37
FHS	-2.06	-1.48	2.34	1.79	2.79	2.59	9.71	2.02	0.49	0.44
HS	-2.11	-1.55	2.32	1.78	2.79	2.61	9.73	2.07	0.49	0.44
RM	-1.31	-1.24	1.67	1.62	2.45	2.49	8.40	1.95	0.43	0.42
GARCH	-1.36	-1.30	1.52	1.47	2.39	2.45	8.72	1.93	0.42	0.41
GEV	-3.74	-3.86	4.02	4.15	3.99	4.15	12.37	2.95	0.71	0.70
GPD	-3.62	-3.74	3.92	4.06	3.90	4.06	11.87	2.92	0.69	0.69
FGEV	-3.67	-3.74	4.02	4.07	3.95	4.05	12.31	2.91	0.70	0.68
FGPD	-3.56	-3.64	3.93	4.01	3.87	3.99	11.84	2.91	0.68	0.67
CAViaR	-1.41	-1.47	2.53	2.68	2.83	2.94	9.18	1.98	0.50	0.50

Table 6: Evaluations of Value-at-Risk for Long Position (Window Size = 250)

In Panel A, (A) the empirical coverage probability, (C) market risk capital requirement and (D) predictive quantile loss are evaluated by averaging 30 companies. The (B) Basel penalty zone is evaluated by the frequency of how many companies fall in each zone. Five tests, (E)–(I), are evaluated by the rejection frequency at the given 10% significance level so that the model with low rejection frequency is preferred by the regulator. Panel B evaluates the portfolio constructed by the equal weighted combination of 30 companies. The (B) Basel penalty zone is evaluated by the colour of the zone. Green is the most acceptable one whilst red is the worst one. *, ** and *** denote that the test statistic is rejected at the 1%, 5% and 10% significance level, respectively. In both panels, the figures in $(\cdot)$ denote the asymptotic distribution for each test statistic. Furthermore, all quantitative measures and statistics for each company and portfolio are calculated using 2,011 VaR forecasts (except Hewlett-Packard Company , AT&T and Verizon Communication Inc. are calculated using 1,416, 2,004 and 1,885 VaR forecasts, respectively).

VaR Models	FAR	HS	FHS	RM	GARCH	GEV	GPD	FGEV	FGPD	CAViaR
Panel A: 30 Components of Dow Jones Industry										
(A) Empirical Coverage Probability	1.07%	1.19%	1.10%	1.82%	1.60%	0.54%	0.55%	0.49%	0.51%	2.83%
(B) Basel Penalty Zone										
Green	30	30	30	7	20	30	30	30	30	0
Yellow	0	0	0	23	10	0	0	0	0	27
Red	0	0	0	0	0	0	0	0	0	3
(C) Market Risk Capital Requirement	40.80%	45.28%	45.29%	39.59%	39.78%	58.09%	60.75%	58.22%	60.21%	47.43%
(D) Predictive Quantile Loss	6.76%	7.44%	6.87%	6.89%	6.80%	7.99%	8.21%	7.61%	7.82%	8.25%
(E) Unconditional Coverage Test (χ_1^2)	7	3	0	27	20	23	21	28	26	30
(F) Independence Test (χ_1^2)	2	9	4	3	4	4	4	6	6	10
(G) Conditional Coverage Test (χ_2^2)	4	12	1	25	19	18	19	19	20	30
(H) Dynamic Quantile Test 1 (χ_5^2)	10	24	10	28	22	14	14	10	9	30
(I) Dynamic Quantile Test 2 (χ_6^2)	11	24	12	29	22	13	13	8	9	30
Panel B: Portfolio										
(A) Empirical Coverage Probability	0.80%	1.24%	1.14%	1.79%	1.84%	0.55%	0.45%	0.55%	0.65%	2.09%
(B) Basel Penalty Zone	Green	Green	Green	Yellow	Yellow	Green	Green	Green	Green	Yellow
(C) Market Risk Capital Requirement	28.84%	27.35%	25.89%	24.64%	25.00%	34.99%	34.54%	35.47%	35.56%	26.66%
(D) Predictive Quantile Loss	4.44%	4.61%	4.48%	4.38%	4.42%	5.05%	4.98%	5.09%	5.18%	4.59%
(E) Unconditional Coverage Test (χ_1^2)	0.91	1.11	0.40	10.27***	11.48***	4.99**	7.81***	4.99**	2.90*	18.32***
(F) Independence Test (χ_1^2)	0.26	0.63	1.24	0.17	1.77	0.12	0.08	0.12	0.17	3.36*
(G) Conditional Coverage Test (χ_2^2)	1.17	1.75	1.65	10.46***	13.27**	5.10*	7.88**	5.10*	3.06	21.70***
(H) Dynamic Quantile Test 1 (χ_5^2)	1.26	7.51	5.85	34.50***	30.31***	12.53**	16.81***	4.28	2.74	78.43***
(I) Dynamic Quantile Test 2 (χ_6^2)	1.38	8.24	9.41	41.20***	38.10***	12.62**	17.26***	5.49	4.59	107.52***

Table 7: Evaluations of Value-at-Risk for Short Position (Window Size = 250)

In Panel A, (A) the empirical coverage probability, (C) market risk capital requirement and (D) predictive quantile loss are evaluated by averaging 30 companies. The (B) Basel penalty zone is evaluated by the frequency of companies' falling into each zone. Five tests, (E)–(I), are evaluated by the rejection frequency at the given 10% significance level so that the model with a low rejection frequency is preferred by the regulator. Panel B evaluates the portfolio constructed by the equal weighted combination of 30 companies. The (B) Basel penalty zone is evaluated by the colour of the zone. Green is the most acceptable whilst red is the worst. *, ** and *** denote that the test statistic is rejected at the given 1%, 5% and 10% significance level, respectively. In both panels, the figures in $(\cdot)$ denote the asymptotic distribution for each test statistic. Furthermore, all quantitative measures and statistics for each company and portfolio are calculated using 2,011 VaR forecasts (except Hewlett-Packard Company , AT&T and Verizon Communication Inc., which are calculated using 1,416, 2,004 and 1,885 VaR forecasts, respectively).

VaR Models	FAR	HS	FHS	RM	GARCH	GEV	GPD	FGEV	FGPD	CAViaR
Panel A: 30 Companies										
(A) Empirical Coverage Probability	0.96%	1.20%	1.03%	1.66%	1.46%	0.56%	0.54%	0.48%	0.48%	2.31%
(B) Basel Penalty Zone										
Green	29	29	30	14	21	30	30	30	30	1
Yellow	1	1	0	16	9	0	0	0	0	28
Red	0	0	0	0	0	0	0	0	0	1
(C) Market Risk Capital Requirement	41.20%	45.13%	42.88%	38.64%	39.04%	55.40%	56.51%	52.53%	53.77%	44.13%
(D) Predictive Quantile Loss	5.93%	7.06%	6.05%	6.09%	5.97%	7.44%	7.51%	6.60%	6.73%	7.10%
(E) Unconditional Coverage Test (χ_1^2)	5	3	0	23	14	23	22	27	28	30
(F) Independence Test (χ_1^2)	1	3	1	1	0	3	4	2	2	4
(G) Conditional Coverage Test (χ_2^2)	4	7	0	19	12	13	18	20	24	30
(H) Dynamic Quantile Test 1 (χ_5^2)	4	16	1	20	11	11	9	5	4	30
(I) Dynamic Quantile Test 2 (χ_6^2)	6	18	3	19	9	10	9	5	9	30
Panel B: Portfolio of 30 Companies										
(A) Empirical Coverage Probability	0.99%	1.79%	1.14%	1.34%	1.09%	0.70%	0.75%	0.35%	0.25%	1.59%
(B) Basel Penalty Zone	Green	Yellow	Green	Green	Green	Green	Green	Green	Green	Yellow
(C) Market Risk Capital Requirement	27.92%	30.31%	25.19%	24.42%	23.98%	33.44%	33.50%	28.62%	28.38%	24.77%
(D) Predictive Quantile Loss	3.65%	4.56%	3.35%	3.34%	3.31%	4.49%	4.43%	3.50%	3.50%	3.59%
(E) Unconditional Coverage Test (χ_1^2)	0.00	10.27***	0.40	2.15	0.17	2.10	1.44	11.53***	16.42***	6.02**
(F) Independence Test (χ_1^2)	0.40	0.17	0.53	0.74	0.49	2.97*	2.71*	0.05	0.02	1.04
(G) Conditional Coverage Test (χ_2^2)	0.40	10.46***	0.94	2.90	0.66	5.06*	4.14	11.57***	16.43***	7.07**
(H) Dynamic Quantile Test 1 (χ_5^2)	0.84	19.96***	3.73	7.23	6.09	14.46**	12.74**	8.63	11.44**	10.44*
(I) Dynamic Quantile Test 2 (χ_6^2)	1.38	8.24	9.41	41.20***	38.10***	12.62**	17.26***	5.49	4.59	107.52***

Table 8: Evaluations of Value-at-Risk for Long Position (Window Size = 500)

In Panel A, (A) the empirical coverage probability, (C) market risk capital requirement and (D) predictive quantile loss are evaluated by averaging 30 companies. The (B) Basel penalty zone is evaluated by the frequency of companies' falling into each zone. Five tests, (E)–(I), are evaluated by the rejection frequency at the given 10% significance level so that the model with low rejection frequency is preferred by the regulator. Panel B evaluates the portfolio constructed by the equal weighted combination of 30 companies. The (B) Basel penalty zone is evaluated by the colour of the zone. Green is the most acceptable one whilst red is the worst one. *, ** and *** denote that the test statistic is rejected at the given 1%, 5% and 10% significance level, respectively. In both panels, the figures in $(\cdot)$ denote the asymptotic distribution for each test statistic. Furthermore, all quantitative measures and statistics for each company and portfolio are calculated using 2,011 VaR forecasts (except Hewlett-Packard Company , AT&T and Verizon Communication Inc., which are calculated using 1,416, 2,004 and 1,885 VaR forecast, respectively).

VaR Models	FAR	HS	FHS	RM	GARCH	GEV	GPD	FGEV	FGPD	CAViaR
Panel A: 30 Companies										
(A) Empirical Coverage Probability	1.05%	1.55%	1.13%	1.82%	1.61%	0.29%	0.34%	0.20%	0.21%	1.76%
(B) Basel Penalty Zone										
Green	29	26	30	10	20	30	30	30	30	11
Yellow	1	4	0	20	10	0	0	0	0	19
Red	0	0	0	0	0	0	0	0	0	0
(C) Market Risk Capital Requirement	38.77%	42.09%	40.88%	36.84%	37.25%	77.46%	74.37%	70.93%	72.42%	42.12%
(D) Predictive Quantile Loss	6.59%	7.85%	6.37%	6.57%	6.46%	9.51%	9.39%	8.75%	8.92%	6.88%
(E) Unconditional Coverage Test (χ_1^2)	5	13	0	27	18	30	28	29	30	26
(F) Independence Test (χ_1^2)	3	10	2	3	4	5	6	1	2	7
(G) Conditional Coverage Test (χ_2^2)	6	18	2	22	16	28	28	29	30	22
(H) Dynamic Quantile Test 1 (χ_5^2)	9	27	6	25	20	19	16	15	16	24
(I) Dynamic Quantile Test 2 (χ_6^2)	11	27	7	25	21	17	9	10	9	30
Panel B: Portfolio of 30 Companies										
(A) Empirical Coverage Probability	0.62%	1.70%	0.85%	1.70%	1.64%	0.34%	0.28%	0.23%	0.23%	1.19%
(B) Basel Penalty Zone	Green	Green	Green	Green	Yellow	Green	Green	Green	Green	Green
(C) Market Risk Capital Requirement	27.84%	27.23%	24.23%	22.18%	23.12%	42.30%	38.84%	39.69%	42.00%	24.13%
(D) Predictive Quantile Loss	3.67%	4.75%	3.49%	3.52%	3.50%	5.64%	5.37%	5.29%	5.77%	3.63%
(E) Unconditional Coverage Test (χ_1^2)	2.91*	7.24***	0.42	7.24***	6.20**	10.40***	12.75***	15.50***	15.50***	0.61
(F) Independence Test (χ_1^2)	0.14	1.04	2.48	0.38	2.85*	0.04	0.03	0.02	0.02	0.51
(G) Conditional Coverage Test (χ_2^2)	3.04	8.30**	2.89	7.64**	9.07**	10.43***	12.76***	15.50***	15.50***	1.12
(H) Dynamic Quantile Test 1 (χ_5^2)	2.67	46.35***	5.94	19.09***	26.09***	24.50***	29.72***	10.61*	10.61*	4.28
(I) Dynamic Quantile Test 2 (χ_6^2)	2.96	47.31***	7.61	23.22***	29.94***	24.57***	30.08***	10.81*	10.85*	7.53

Table 9: Evaluations of Value-at-Risk for Short Position (Window Size = 500)

In Panel A, (A) the empirical coverage probability, (C) market risk capital requirement and (D) predictive quantile loss are evaluated by averaging 30 companies. The (B) Basel penalty zone is evaluated by the frequency of companies' falling into each zone. Five tests, (E)–(I), are evaluated by the rejection frequency at the given 10% significance level so that the model with low rejection frequency is preferred by the regulator. Panel B evaluates the portfolio constructed by an equal weighted combination of 30 companies. The (B) Basel penalty zone is evaluated by the colour of the zone. Green is the most acceptable one whilst red is the worst one. *, ** and *** denote that the test statistic is rejected at the given 1%, 5% and 10% significance level, respectively. In both panels, the figures in $(\cdot)$ denote the asymptotic distribution for each test statistic. Furthermore, all quantitative measures and statistics for each company and portfolio are calculated using 2,011 VaR forecasts (except Hewlett-Packard Company , AT&T and Verizon Communication Inc., which are calculated using 1,416, 2,004 and 1,885 VaR forecast, respectively).

VaR Models	FAR	HS	FHS	RM	GARCH	GEV	GPD	FGEV	FGPD	CAViaR
Panel A: 30 Companies										
(A) Empirical Coverage Probability	0.99%	1.49%	1.10%	1.72%	1.44%	0.33%	0.39%	0.22%	0.23%	1.67%
(B) Basel Penalty Zone										
Green	29	27	30	11	23	30	30	30	30	14
Yellow	1	3	0	19	7	0	0	0	0	16
Red	0	0	0	0	0	0	0	0	0	0
(C) Market Risk Capital Requirement	38.69%	43.70%	40.20%	36.38%	36.75%	71.05%	68.11%	61.35%	62.18%	39.90%
(D) Predictive Quantile Loss	5.79%	7.62%	5.78%	5.96%	5.76%	8.70%	8.55%	7.30%	7.36%	6.32%
(E) Unconditional Coverage Test (χ_1^2)	3	11	1	24	11	27	25	30	30	19
(F) Independence Test (χ_1^2)	0	6	1	1	1	1	4	1	0	1
(G) Conditional Coverage Test (χ_2^2)	3	11	1	21	8	27	24	29	29	16
(H) Dynamic Quantile Test 1 (χ_5^2)	4	22	1	19	7	9	11	20	14	19
(I) Dynamic Quantile Test 2 (χ_6^2)	7	24	1	19	7	6	9	9	9	21
Panel B: Portfolio of 30 Companies										
(A) Empirical Coverage Probability	0.91%	2.10%	1.13%	1.47%	1.08%	0.17%	0.68%	0.11%	0.28%	1.25%
(B) Basel Penalty Zone	Green	Yellow	Green	Green	Green	Green	Green	Green	Green	Green
(C) Market Risk Capital Requirement	27.17%	29.64%	22.96%	22.49%	22.82%	43.74%	38.90%	28.88%	28.65%	22.44%
(D) Predictive Quantile Loss	3.65%	5.20%	3.13%	3.22%	3.14%	5.23%	5.00%	3.48%	3.45%	3.21%
(E) Unconditional Coverage Test (χ_1^2)	0.16	16.33***	0.31	3.50*	0.10	18.76***	2.05	22.69***	12.75***	1.01
(F) Independence Test (χ_1^2)	0.29	1.42	0.46	0.78	0.41	0.01	3.32*	0.00	0.03	0.56
(G) Conditional Coverage Test (χ_2^2)	0.45	17.78***	0.77	4.29	0.52	18.75***	5.36*	22.68***	12.76***	1.58
(H) Dynamic Quantile Test 1 (χ_5^2)	0.70	33.36***	4.13	9.21	4.12	12.23**	17.06***	13.96**	9.12	4.57
(I) Dynamic Quantile Test 2 (χ_6^2)	1.23	36.25***	5.60	10.70*	8.68	12.54*	17.67***	13.97**	9.53	5.22