Private Equity Fund Fees: Valuation and Incentives*

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Abstract

Compensation of private equity fund managers typically consists of a fixed management fee and a performance-related carried interest that entitles them to option-like payoffs. We develop a comprehensive model to value carried interest based on risk-neutral pricing techniques. In this model, we assume that investors and managers display constant relative risk aversion (CRRA) and adopt an equilibrium framework in which investors earn zero abnormal returns. This model allows to study incentive effects. The results show that managers have an incentive for excessive risk-taking in case they only consider fee income from their current fund. We also extend the model to the more realistic setting in which managers take into account potential fee income from follow-on funds. In this case risk-taking incentives depend on the managers level of skill, where highly skilled mangers even have an incentive to damp down risk. Our model also provides a number of other interesting results, including a possible explanation for the existence of the option-like contract that is standard in the private equity industry.

Keywords: private equity, buyout funds, fund fees, Monte Carlo simulation, risk-neutral pricing

JEL Classification: G1, G1

Compensation of funds managers increasingly involves elements of profit sharing that entitle managers to option-like payoffs that are contingent on the performance of the managed fund. An important example is the compensation of private equity fund managers. While the literature has spent much effort understanding some aspects of the private equity market, it is very surprising that there is no clear answers to the basic questions of the rationale of this compensation structure and its impact on the funds' choices of investments. How can the performance related compensation of private equity fund managers be valued and what is its value relative to fixed compensation? What are the determinants of private equity fee value and how can fund managers influence value? Does the typical compensation with convex payoffs increase the managers' incentive to take risks? Is there a rationale for investors to entitle managers with convex compensation structures instead of giving them linear contracts? This paper aims to shed light on these questions.

Private equity funds are typically organized as limited partnerships, with private equity firms serving as General Partners (GPs) of the funds, and large institutional investors and wealthy individuals providing bulk of the capital known as Limited Partners (LPs). These partnerships typically last for 10 years, and legal agreements signed at a fund initiation define the expected payments to GPs. These payments consist of a fixed component (called "management fee") and a performance related component (called "carried interest"). The fixed management fee resembles well pricing terms of mutual and hedge funds, while the performance related carried interest is uncommon among most mutual funds and is different from the variable incentive fees of hedge funds.[1]

Carried interest is an option-like position on the total proceeds of all investments of the fund. In practice, multiple, staggered investments and stepwise liquidation of the investments make carried interest considerably more complex than a simple call option on a traded asset. In particular, the option payoffs are typically contingent upon the

[1]For an overview on the typical compensation structure of mutual fund refer to Chordia (1996) or Christoffersen (2001); for hedge funds refer to Goetzmann et al. (2003), Hodder and Jackwerth (2007), or Agarwal et al. (2009).

performance of the fund, where performance is typically measured by the internal rate of return of the fund cash flows. To derive solutions to the problem of valuing these option-like positions, we employ a continuous-time model of the cash flow dynamics of private equity funds. This model consists of three main components which correspond to the three essential phases of the private equity fund lifecycle: the drawdowns from the committed capital paid into the fund; the performances of the investments effected by the fund; and the distributions of dividends and proceeds taken out of the fund.

A risk-neutral valuation framework is applied to determine the value of the carried-interest option. Since we are dealing in an incomplete market setting, we have to impose some assumption on the preferences of the agents in our model. We assume that GPs and LPs have power utility displaying constant relative risk aversion (CRRA). To implement the risk-neutral valuation, we also have to impose some restriction on the expected rate of return of a fund to prevent the ex-ante net present value of the LPs' stake in a fund from turning negative. We assume that the ex-ante required rate of return of a fund is determined in a partial-equilibrium framework that is qualitatively similar to that considered in Berk and Green (2004) in the context of mutual funds. In this framework, LPs supply their capital competitively to GPs whose skill is a scarce resource which results in net of fee expected excess returns of the LPs being equal zero in equilibrium.[2] This equilibrium framework also allows us to infer the abnormal return GPs have to generate in order to compensate LPs for the fees taken.

In the application of our model, we calibrate it to data of buyout funds. Our valuation results reveal that present value of lifetime fee revenues of the GPs of a typical buyout fund amounts to around 20 percent of committed capital. Management fees account for the largest portion of the total fee value. Around four-fifth of the value of the total fee revenues derive from the fixed management fee and only around one-fifth from the variable revenue generated by the carried interest. In addition, our results show

[2]This is consistent with the empirical results of e.g. Jones and Rhodes-Kropf (2003) and Jegadeesh et al. (2009) that show that net of fee abnormal fund returns are close to zero.

that GPs have to generate substantial abnormal returns to compensate LPs for the fees taken. Given the typical fee structure of buyout funds, our results imply that gross of fee abnormal returns amount to around 6.5 percent p.a.

We also employ our model to analyze incentives generated by the typical compensation scheme of private equity funds. We first consider the case of a GP that aims to maximize fee income from a single fund only. Consistent with standard option pricing theory, our results suggest that the option-like compensation contract encourages excessive risk taking by the GPs, particularly if they display a low relative risk aversion.[3]

In a second step, we extend our analysis to a more realistic setting in which GPs do consider not only fee compensation from their current fund, but also potential subsequent compensation based on follow-on funds. Our results highlight that the pressure to raise follow-on funds creates a unique incentive structure that is very different from the single fund case. We show that risk-taking behavior then depends on the GPs level of skill, i.e. their ability to generate high abnormal returns. Low skilled GPs still have an incentive for excessive risk taking, while high skilled investors even have an incentive to decrease risk. The rationale behind this is that GPs are essentially equipped with two kinds of option-like contracts in the case they aim to raise multiple funds in sequence. The first option is again the carry option. The second one is to receive compensation from follow-on funds which shares some characteristics with a binary cash-or-nothing call on the performance of the current fund. This second option is not necessarily an increasing function in asset volatility. Rather, its sensitivity to volatility depends on the abnormal returns GPs can generate. This results in total fee values than can increase or decrease with asset volatility, depending on the level of skill of the GPs. We also address the question why this complex incentive structure is necessary in the private equity segment to limit risk-taking. We argue that the most important reason has to do with agency problems between. Usually, LPs have very limited abilities to observe and limit GPs

[3]The general problem that convex compensation schemes can increase asset managers' risk-taking has initially been challenged by Carpenter (2000) and Ross (2004).

risk-taking. Therefore, some external mechanics must be installed that limits GPs risk appetite. This is achieved by the pressure to raise follow-on funds which results from the finite lifetime of the funds. However, we also show that this mechanism does not work perfect because it still leaves low skilled GPs with an incentive to gamble.

In a third step, we address incentives of the GPs to alter risk-taking during the life of a fund. In a general context, Carpenter (2000) points out that an option-like compensation encourages managers to dynamically adjust volatility as the asset value changes. This problem can even be gravitated in the private equity area because past fund performance both impacts the value of the carry option and the option to receive compensation from follow-on funds. However, we show that the resulting incentive problems are mostly resolved in standard private equity contracts because GPs risk-shifting ability is limited by the fact that they are only allowed to draw down capital over a limited commitment period and are not allowed to reinvest proceeds of a fund. This also gives an interesting explanation for these seemingly odd contractual arrangements as a means of limiting risk-shifting behavior of fund managers during the lifetime of a fund.

In a final step, our analysis gives an interesting explanation for the option-like compensation contract that is standard in the private equity industry. We show that that the contract encourages GPs to search for investments that have a low correlation with the market and to run funds with a high level of idiosyncratic risk. This is reasonable for two reasons. First, LPs typically seek additional diversification for their overall portfolios, not exposure to market risk. Second, it prevents that GPs have an incentive to spend too little time with any one investment in the attempt to manage more projects.

This paper is related to the literature that investigates fee terms of private equity partnerships. The first paper to address this issue is Sahlman (1990), who provides a broad overview of the compensation structure of venture capital contracts. Fenn et al. (1997) expand the discussion to non-venture limited partnerships. Gompers and Lerner (1999a) explore cross-sectional and time-series variation in the terms of venture capital funds. They find that GPs at young venture funds tend to receive higher management

fees (in percentage terms) and lower carry, which they take as evidence of a model in which limited partners learn about the ability of GPs to create returns. They argue that GPs at young firms need less carry to exert effort because they will want to work hard to demonstrate their ability. Similar issues are addressed by Litvak (2004), who extends the analysis of Gompers and Lerner (1999a) from a more legal perspective. Neither of these papers uses an option-pricing framework to derive the value of carried interest. An exception to this is Metrick and Yasuda (2010), who also employ a risk-neutral valuation framework to estimate the value of various pricing terms. However, they do not develop a comprehensive model for the cash flow and value dynamics of private equity funds, as done in this paper. This framework allows us to analyze how the value of the pricing terms is related to the cash flow dynamics and the risk/return characteristics of a fund. This gives rise to several important determinants of fund fee value that have not previously been studied. In addition, our analysis goes beyond that of Metrick and Yasuda (2010) in that we also study incentives generated by the typical compensation schemes of private equity funds.

We also add to the literature that tries to explain the financial structure of private equity funds, e.g. Axelson et al. (2009). Our main contribution is to show that a fund's finite lifetime, its defined commitment period and the fact that a reinvestment of proceeds is not allowed act as external mechanisms that limit risk-taking. We also contribute to the literature focusing on the investment behavior of private equity funds. Consistent with the empirical results of Ljungqvist et al. (2008), our model can, for example, explain why inexperienced GPs carry out riskier investments.

The remainder of this paper is organized as follows. In the next section, we outline the institutional setting and define fee components. Section 2 presents our model for the fund dynamics and fees. Section 3 illustrates the risk-neutral valuation framework for private equity fund fees. Section 4 highlights the results of a model calibration, analyzes fee value determinants and studies incentive effects of the standard private equity compensation structure. The paper concludes with Section 5.

1 Institutional Setting and Fund Fees

Investments in private equity are typically intermediated through private equity funds. Thereby, a private equity fund is organized as a limited partnership in which the private equity firms serve as General Partners (GPs). The investors, called Limited Partners (LPs), commit to provide a certain amount of capital to the private equity fund, the so-called committed capital. The GP then has an agreed time period in which to invest committed capital which is commonly referred to as the fund commitment period. When a GP identifies an investment opportunity, he "calls" money at any time during the pre-specified commitment period from the LPs up to the amount committed. Capital calls, also called drawdowns or takedowns, usually occur unscheduled over the commitment period, where the exact timing only depends on the investment decisions of the GPs. Once drawdowns occur, the available cash is immediately invested in managed assets and the size of the portfolio begins to increase. The GP has an agreed time period in which to return capital to the LPs, usually ten to fourteen years, which is called the total legal lifetime of the fund. When an investment is liquidated, the GP distributes the proceeds either in marketable securities or in cash.[4]

Following the typical structure of private equity funds, the GPs receive two types of compensations for managing the investments: a fixed component (called "management fee") and a performance related component (called "carried interest" or simply "carry").[5]

The management fee is generally expressed as a percentage of the committed cap-

[4]For a more thorough introduction on the organizational structure of private equity funds, for example, refer to see Sahlman (1990), Fenn et al. (1997), Gompers and Lerner (1999b), or Phalippou (2007).

[5]Besides management fee and carried interest, private equity funds also often receive transaction and monitoring fees. Both types of additional fees are more common for buyout than for venture capital funds. A transaction fee is typically charged by buyout funds to their LPs when buying or selling a portfolio company. Additionally, many buyout funds charge monitoring fees to their portfolio companies. This is to compensate the fund for the time and effort spent by the GPs in advising and monitoring the portfolio companies. In most cases, these fees are shared with the LPs that receive 80 percent of the monitoring fees and GPs that receive 20 percent. Transaction and monitoring fees are not covered in the analysis of this paper and the interested reader is referred to Metrick and Yasuda (2010) for a more detailed overview.

ital and paid annually. The most common fees are around 2 percent yearly and vary based on fund sizes (see Metrick and Yasuda (2010)). Often the fees decrease after the commitment period to reflect the fact that less time is required in managing activities, especially when investments are mature or partly realized. Tapering the management fee is effected by either reducing the accounting basis on which the rate is applied -or through reduction of the percentage rate paid for compensation. Tapered fee corresponds to the real purpose of the management fee to cover the costs of running and administering the fund. For example, if a fund charges a fixed 2 percent annual management fees on committed capital for the entire fund lifetime of 10 years, then the lifetime management fees of the fund amount to 20 percent of the committed capital. Thus only the remaining 80 percent of the committed capital will be available for investments into portfolio companied. We will denote this as investment capital of the fund in the following. In general, investment capital is equal to committed capital of a fund minus lifetime management fees.

The second source of compensation is the carried interest or carry, which entitles venture capital firms a certain share of the capital gains from the fund. Typically, the carried interest is only paid if LPs received initial investment plus some other form of pre-specified interests. Most LPs require interests on their capital known as a hurdle rate. This hurdle rate is not a guaranteed interest payment to the fund providers, but rather represents target returns above which venture capital firms are entitled to receive the carried interest. The hurdle rate is typically 8 percent and the carried interest level most commonly employed in the industry is 20 percent (see Metrick and Yasuda (2010)).

[Insert Table 1 about here]

Table 1 illustrates the computation of the carried interest using a simple numerical example. Assume a private equity fund with a committed capital of $100M, a carried interest level of 20 percent, a hurdle rate of 8 percent, and a fixed lifetime of ten years. For simplicity, we assume that there are no management fees and that the committed

capital of the fund is fully drawn by the fund and hence, the absolute value of the negative cash flows during the first four years represents committed capital (i.e. \$100M). The fund has exited some of their portfolio companies beginning of year five and continued to the end of year 10. The table shows that the fund made profits of \$150M net of investments or equivalently a multiple of 2.5 (250/100) at the end of fund life.

The calculated cash flow based internal rates of return (IRR) show that the distribution schedule of the fund satisfies the 8 percent hurdle rate requirement in year seven. One year later, the GPs are entitled to receive carried interest. The exact amount of the carried interest is thereby affected by the existence of a catch-up clause, as shown in Table 1. With no catch-up clause, the GPs receive 20 percent of the capital distributions beginning of year eight and continued to the end of year 10. This results in total carried interest payments of \$22M, which is lower than the carried interest level of 20 percent times the net profits of \$150M of the fund. Thus the GPs receive a lower fraction of the net profits of the fund than the defined carried interest level when no catch-up clause exists. The key idea of a catch-up clause is that the GPs of a fund with a catch-up and hurdle rate receive a fraction of the profits of the fund equal to the carried interest level as long as the fund is sufficiently profitable, i.e., the fund has a IRR that exceeds the hurdle rate. Table 1 shows that the catch-up provision in this example implies that the GPs receive \$18M carried interest in year eight, which is equal to the carried interest level of 20 percent times the net profits of the fund of \$90M (\$190M of distributions on \$100M of committed capital) up to year eight. After year eight, the GPs again receive a fraction of 20 percent of the capital distributions. This results in total carried interest payments of \$30M, which is exactly the carried interest level of 20 percent times the net profits of \$150M of the fund. In this example, we have chosen a catch-up provision of 100 percent. A fund with a catch-up percentage below 100 percent will still receive a proportion equal to the carried interest level of the fund profits, but at a somewhat slower pace than in the example shown here.

The above description has made clear that carried interest is an option-like position

on the total proceeds of the fund investments. Valuing such a contingent claim first requires modeling the cash flow dynamics of private equity funds.

2 The Model

2.1 Fund Dynamics

Our model consists of three main components which correspond to the essential phases of the private equity fund lifecycle: the drawdowns from the committed capital paid into the fund; the performances of the investments effected by the fund; and the distributions of dividends and investment proceeds taken out of the fund.

A. Capital Drawdowns

We consider a fund that has total initial commitments from the LPs given by C. If MF_{T_l} denotes cumulated lifetime management fees, the fund investment capital I is given by $I = C - MF_{T_l}$. The fund has a legal maturity T_l and a commitment period T_c, where $T_c \leq T_l$. Cumulated capital drawdowns up to time t are denoted by D_t. We assume capital to be drawn over the commitment period T_c at some non-negative rate δ_t from the remaining undrawn investment capital $U_t = I - D_t$. The dynamics of the cumulated capital drawdowns D_t can then be described by

$$dD_t = \delta_t U_t 1_{\{0 \leq t \leq T_c\}} dt, \tag{2.1}$$

where $\delta_t \geq 0$ denotes the fund's drawdown rate and $1_{\{.\}}$ is an indicator function.

Equation (2.1) accounts for the fact that after high initial investment activity, drawdowns of private equity funds are typically carried out at a declining rate, as fewer new investments are made, and follow-on investments are spread out over a number of years.

Under the specification (2.1), cumulated capital drawdowns D_t are given by

$$D_t = I\left[1 - \exp\left(-\int_0^{t\leq T_c} \delta_u du\right)\right]. \tag{2.2}$$

This shows that the initially high capital drawdowns at the start of the fund decrease over the commitment period T_c of the fund, as undrawn investment capital, $U_t = I\exp\left(-\int_0^{t\leq T_c} \delta_u du\right)$, decays exponentially over time. Furthermore, equation (2.2) has the reasonable feature that cumulated drawdowns D_t can never exceed the total investment capital I, i.e., $D_t \leq IC$ for all $t \in [0, T_c]$. At the same time the model also allows for a certain fraction of the investment capital I not to be drawn, as the commitment period T_c acts as a cut-off point for capital drawdowns.

As investment opportunities typically do not arise constantly over the commitment period T_c, we introduce a stochastic process for the drawdown rate δ_t. Its specification under the objective probability measure $\mathbb{P}$ given by

$$d\delta_t = \kappa_\delta(\theta_\delta - \delta_t)dt + \sigma_\delta\sqrt{\delta_t}dB_{\delta,t}, \tag{2.3}$$

where $\theta_\delta > 0$ is the long-run mean of the drawdown rate, $\kappa_\delta > 0$ governs the rate of reversion to this mean, $\sigma_\delta > 0$ reflects the volatility of the drawdown rate, and $B_{\delta,t}$ is a standard Brownian motion.

The drawdown rate behavior implied by the above square-root diffusion ensures that negative values of the drawdown rate are precluded[6] and that the drawdown rate randomly fluctuates around some mean level θ_δ.

B. Capital Distributions

As capital drawdowns occur, the available capital is immediately invested in managed

[6] As we model capital distributions and capital drawdowns separately, we have to restrict capital drawdowns to be strictly non-negative at any time t during the period $[0, T_c]$. The square-root diffusion was initially proposed by Cox et al. (1985) as a model of the short rate and is frequently denoted as CIR model. If $\kappa_\delta, \theta_\delta > 0$, then δ_t will never be negative. If $2\kappa_\delta\theta_\delta \geq \sigma_\delta^2$, then δ_t remains strictly positive for all t, almost surely. See Cox et al. (1985), p. 391.

assets and the portfolio of the fund begins to accumulate. As the underlying investments of the fund are gradually exited, cash or marketable securities are received and finally returns and proceeds are distributed to the LPs of the fund. Let cumulated capital distributions up to some time t during the legal lifetime T_l of the fund be denoted by R_t. Recognizing that the size and timing of repayments are based on the performance of the fund, it is assumed that capital distributions occur at a non-negative rate ρ_t from the total investment portfolio value V_t of the fund at time t. The dynamics of the cumulated capital distributions R_t can then be described by

$$dR_t = \rho_t V_t dt, \quad \text{if} \quad t < T_l, \quad \text{and} \quad R_t = \int_0^t \rho_u V_u du + V_t 1_{\{t=T_l\}}, \tag{2.4}$$

where $\rho_t \geq 0$ denotes the fund's distribution rate.

The ordinary differential equation, $dR_t = \rho_t V_t dt$, illustrates that capital repayments occur at the non-negative distribution rate ρ_t from the current portfolio value V_t. This, however, holds only for $t < T_l$. As funds are fully liquidated at the end of their legal lifetime, cumulated capital distributions over the entire life of the fund must also include the final reimbursement of the assets of the fund at maturity T_l, i.e., $R_{T_l} = \int_0^{T_l} \rho_u V_u du + V_{T_l}$.

To capture the erratic feature of real world private equity fund capital distribution, we also introduce a stochastic process for the distributions rate ρ_t. The specification under the objective probability measure $\mathbb{P}$ is given by

$$d\rho_t = \kappa_\rho(\theta_\rho - \rho_t)dt + \sigma_\rho \sqrt{\rho_t} dB_{\rho,t}, \tag{2.5}$$

where $\theta_\rho > 0$ is the long-run mean of the distribution rate, $\kappa_\rho > 0$ governs the rate of reversion to this mean, $\sigma_\rho > 0$ reflects the volatility of the distribution rate, and $B_{\rho,t}$ is a second standard Brownian motion.

We model distributions and drawdowns separately and, therefore, must similarly

restrict capital distributions to be strictly non-negative at any time t during the legal lifetime T_l of the fund. This is again achieved by assuming a square-root diffusion for the dynamics of the distributions rate.

C. Investment Value

The last step in the modeling of private equity funds is characterizing the dynamics of fund value. A natural specification for the value dynamics V_t under the objective probability measure $\mathbb{P}$ is the stochastic differential equation

$$dV_t = V_t(\mu_V dt + \sigma_V dB_{V,t}) + dD_t - dR_t, \tag{2.6}$$

where μ_V is the (gross of fee) mean rate of return, $\sigma_V > 0$ is the constant return volatility, and $B_{V,t}$ is a third standard Brownian motion.

Equation (2.6) shows that the change in value of the fund is made of the performance of the existing investments V_t that are assumed to obey normally distributed returns. Besides, the value of the fund is augmented by the capital drawdowns and decreased by the capital distributions.

Substituting the definition of dR_t, equation (2.6) can be simplified to

$$dV_t = (\mu_V - \rho_t)V_t dt + \sigma_V V_t dB_{3,t} + dD_t, \quad \text{if } t < T_l, \tag{2.7}$$

which is the stochastic differential equation of a so-called inverse gamma process that turns into a standard geometric Brownian motion here in case of $t > T_c$.[7] Note here that final value is $V_{T_l} = 0$ because the final cash flow repayment inherent in specification (2.4). In addition, note that also $V_0 = 0$ holds in the model by definition.

[7] This follows as $dD_t = 0$ for $t > T_c$ hold by definition.

2.2 Manager Compensation

GPs typically receive two forms of compensations for managing the partnership interests. The first, is fixed revenue and paid as management fees, while the second is performance-based revenue and payable as carried interest.

The formal definition of the management fees is straightforward. Let MF_t denote cumulated management fees up to some time $t \in [0, T_l]$. If management fees are defined as a percentage c_{mf} of the committed capital C and paid continuously, the dynamics of the management fees can be represented by the ordinary differential equation

$$dMF_t = c_{mf} C dt. \tag{2.8}$$

The definition of the carried interest is more complex as this performance based compensation depends on the fund cash flows in a nontrivial manner. Let CI_t denotes the cumulated carried interest up to some time $t \in [0, T_l]$. The carried interest entitles the GPs a certain share of the capital gains from the fund. If the carried interest level is given by c_{ci}, the GPs receive the fraction c_{ci} of the net cash flows of the fund after management fees. Mathematically, the dynamics of the carried interest can be described by

$$dCI_t = c_{ci} \max\{dR_t - dD_t - dMF_t, 0\} 1_{\{IRR_t > h\}}, \tag{2.9}$$

where taking the maximum of the net cash flows after management fees and zero, $\max\{dR_t - dD_t - dMF_t, 0\}$, makes sure that instantaneous carried interest payments can never become negative. In addition, multiplying by the indicator function $1_{\{IRR_t > h\}}$ makes sure that the performance related carried interest is only payable at time t if the internal rate of return of the fund at that time, IRR_t, exceeds the specified hurdle rate h, i.e. $IRR_t > h$ holds. The internal rate of return is the performance measure commonly employed in the private equity industry to evaluate the return of a fund.

In continuous-time, the internal rate of return IRR_t of the net fund cash flows after management fees at time t can be found as a solution to[8]

$$\int_0^t e^{-IRR_t u}(dR_u - dD_u - dMF_u) = 0. \tag{2.10}$$

Equation (2.9) implicitly assumes that the carried interest is paid without the existence of a catch-up clause. If the carried interest is paid with a 100 percent catch-up provision, the dynamics of the carried interest become somewhat more complex. Again, carried interest is only payable at time t if $IRR_t > h$ holds. In addition, if the fraction of the cumulated carried interest of the net profits of the fund is equal to the carried interest level, i.e. $CI_t/(R_t - C) = c_{ci}$ holds, the GPs receive an instantaneous carried interest of $c_{ci} \max\{dR_t - dD_t - dMF_t, 0\}$. This is similar to the definition of the carried interest without a catch-up in equation (2.9). In contrast, if $CI_t/(R_t - C) < c_{ci}$ holds, instantaneous carried interest payments are equal to $\min\{c_{ci}(R_t - C) - CI_t, dR_t - dD_t - dMF_t\}$. Thereby, taking the minimum here makes sure that the catch-up payments, $c_{ci}(R_t - C) - CI_t$, cannot exceed instantaneous net fund cash flows after management fees, $dR_t - dD_t - dMF_t$.

3 Fee Valuation

3.1 Ex-Ante Equilibrium Fee Value

We now derive the value of lifetime private equity fund fees using risk-neutral pricing techniques in a partial-equilibrium framework. The value of fund fees V_t^{GP} at time $t \in [0, T_l]$ is defined by the discounted value of all expected outstanding lifetime fund fees, including management fees and the carried interest. Applying a risk-neutral valuation

[8]Note that the internal rate of return here is just the continuously compounded rate of return that makes the net present value of the fund cash flows equal to zero.

approach, the arbitrage-free value of the fund fees is given by

$$V_t^{GP} = V_t^{MF} + V_t^{CI} = E_t^{\mathbb{Q}}\left[\int_t^{T_l} e^{-r_f(u-t)} dMF_u\right] + E_t^{\mathbb{Q}}\left[\int_t^{T_l} e^{-r_f(u-t)} dCI_u\right], \tag{3.1}$$

where $E_t^{\mathbb{Q}}[\cdot]$ denotes the expectations operator conditional on the information set available at time t. This conditional expectations operator is defined under the risk-neutral probability measure $\mathbb{Q}$. Therefore, discounting at the riskless rate r_f is appropriate in equation (3.1). From specification (3.1), the value of lifetime fund fees is made of two components. The first term, V_t^{MF}, is the value of the outstanding management fees. The second term, V_t^{CI}, corresponds to the value of the outstanding carried interest.

The evaluation of the first integral for the value of the management fees is trivial. This is since the fixed management fee is a contractual arrangement that is paid riskless and by a constant amount per time period. Therefore, we can eliminate the first expectation in (3.1), as the expectation of a constant is simply the constant itself.[9] Substituting equation (2.8), the value of the outstanding management fees turns out to be

$$V_t^{MF} = c_{mf} CO\left[\int_t^{T_l} e^{-r_f(u-t)} du\right] = c_{mf} CO \frac{1 - e^{-r_f(T_l - t)}}{r_f}. \tag{3.2}$$

Similarly, substituting equation (2.9), the value of the outstanding carried interest (with no catch-up) can be evaluated by solving[10]

$$V_t^{CI} = E_t^{\mathbb{Q}}\left[\int_t^{T_l} e^{-r_f(u-t)} c_{ci} \max\{dR_u - dD_u - dMF_u, 0\} 1_{\{IRR_u > h\}}\right]. \tag{3.3}$$

From equation (3.3), one can directly infer that the value of the carried interest is a contingent claim on the capital drawdowns and distributions of a private equity fund. The complication here arises from the fact that the state variables underlying the valu-

[9] Note that this relationship holds regardless of whether the expectation is defined under the objective measure $\mathbb{P}$ or under the risk-neutral measure $\mathbb{Q}$.

[10] Note that we can similarly define the value of the outstanding carried interest with a catch-up provision.

ation, i.e. the assumed cash flow processes, do not represent traded assets. In such an incomplete market setting, risk-neutral pricing based on arbitrage considerations alone is not sufficient to determine unique prices of contingent claims. Rather, we have to impose some assumption on the preferences of the GPs and LPs.

To illustrate this point, the risk sources underlying our cash flows model have to be transformed. Applying Girsanov's Theorem, as for example outlined in Duffie (2001), it follows that the underlying stochastic processes for the drawdown rate, distribution rate, and the investment value under the risk-neutral probability measure $\mathbb{Q}$ are given by

$$d\delta_t = [\kappa_\delta(\theta_\delta - \delta_t) - \lambda_\delta \sigma_\delta \sqrt{\delta_t}]\ dt + \sigma_\delta\ \sqrt{\delta_t}\ dB^{\mathbb{Q}}_{\delta,t}, \tag{3.4}$$

$$d\rho_t = [\kappa_\rho(\theta_\rho - \rho_t) - \lambda_\rho \sigma_\rho \sqrt{\rho_t}]\ dt + \sigma_\rho\ \sqrt{\rho_t}\ dB^{\mathbb{Q}}_{\rho,t}, \tag{3.5}$$

$$dV_t = (\mu_V - \lambda_V \sigma_V) V_t dt + \sigma_V V_t dB^{\mathbb{Q}}_{V,t} - dR_t + dD_t, \tag{3.6}$$

where $B^{\mathbb{Q}}_{1,t}$, $B^{\mathbb{Q}}_{2,t}$ and $B^{\mathbb{Q}}_{3,t}$ are $\mathbb{Q}$-Brownian motions and λ_δ, λ_ρ and λ_V are the corresponding market prices of risk.

To simplify the valuation, we assume that the drawdown and distribution rate carry zero systematic risk. Under this condition the market prices of risk λ_δ and λ_ρ are equal to zero and the drift coefficients of the stochastic processes δ_t and ρ_t are not affected by the change of the probability measure. The third market price of risk λ_V can be derived using a version of the intertemporal capital asset pricing model (ICAPM) of Merton (1973). In this version of the ICAPM the expected return μ_i of an asset i satisfies the relation: $\mu_i - r_f = \eta Cov_{iM}$, where Cov_{iM} is the covariance of the return on asset i with the return of the market (wealth) portfolio M and η is the coefficient of relative risk aversion of a representative agent. We assume that GPs and LPs both have power utility displaying constant relative risk aversion (CRRA). The GPs and LPs coefficients of relative risk aversion are given by η_{GP} and η_{LP}. The carry valuation in equation (3.3) is done from the perspective of the GPs. Therefore, the GPs relative risk aversion η_{GP}

is the relevant one to determine λ_V. Under these conditions, the market price of risk is given by:

$$\lambda_V \equiv \frac{\mu_V - r_f}{\sigma_V} = \eta_{GP}\ Corr_{VM}\ \sigma_M. \tag{3.7}$$

That is, the market price of risk is computed by the product of the relative risk aversion coefficient η_{GP}, the correlation coefficient $Corr_{VM}$ between the fund returns and the market portfolio returns, and the volatility σ_M of the market portfolio returns.

Given the developed modeling setup so far, one complication arises. If all model parameters are fixed exogenously, this can lead to situations in which the (ex-ante) net present value of the LPs' stake in a fund is negative. This is not going to make much sense in equilibrium because investors would agree to pay fees to lose money then. Instead, they will require GPs to add value somewhere to justify their fees. We rule out this possibility by assuming that the ex-ante expected rate of return μ is defined in a partial-equilibrium framework that is outlined in Appendix A. The equilibrium framework is qualitatively similar to that developed in Berk and Green (2004) in the context of mutual funds. The basic idea is that LPs supply their capital competitively but GPs skill is a scarce resource. In this case investors' expected excess returns (net of fees) must equal zero in equilibrium, such that GPs capture all rents.[11] Appendix A shows that this gives rise to the following equilibrium condition

$$E^{\mathbb{Q}}\left[\int_0^{T_l} e^{-r_f u}(dR_u - dD_u - dMF_u - dCI_u)\right] = 0, \tag{3.8}$$

which states that the net present value of the fund's net cash flows after management fees and carried interest payments must be zero at fund initiation in equilibrium. The correct ex-ante equilibrium rate of return can be found by looking for the value of μ^e that makes equation (3.8) to hold. Note that equation must be evaluated from the

[11] Note that this assumption is consistent with the empirical results of Jones and Rhodes-Kropf (2003) and Jegadeesh et al. (2009) that show that net of fees abnormal fund returns are close to zero in the private equity segment.

perspective of the LPs. Therefore, we use the LPs relative risk aversion η_{LP} to compute the market price of risk λ_V here, i.e., $\lambda_V = \eta_{LP}\ Corr_{VM}\ \sigma_M$.

Unfortunately, valuation equation (3.3) and equilibrium condition (3.8) cannot be solved analytically because of the path-dependent structure of the carried interest that is contingent on the internal rate of return of the fund. Therefore, we have to resort to a numerical technique. This is done here by using Monte Carlo simulations based on discrete-time approximations of the stochastic cash flow and fee model. The details of the numerical procedure to solve equations (3.3) and (3.8) are outlined in Appendix B.

3.2 Ex-Ante Equilibrium Abnormal Returns

As pointed out, the fee structure is a cost to the investors for which they will require GPs to add value. The question that naturally emerges then is by how much does a private equity fund's expected rate of return has to exceed that of a passively managed portfolio with the same level of risk in order to justify the fee structure? Our framework allows us to explore this question in a simple but potentially interesting way.

Equilibrium condition (3.8) gives the ex-ante equilibrium expected rate of return μ_V^e of a private equity fund for a given fee structure and level of risk σ_V. The expected rate of return of a passively managed portfolio with the same level of risk can be derived from the version of the ICAPM we also used to determine the market price of risk λ_V. This yields an expected rate of return μ_V commensurate with the risk σ_V of

$$\mu_V = r_f + \eta_{LP}\ Corr_{VM}\sigma_V\sigma_M. \tag{3.9}$$

Note that the relative risk aversion of the LPs η_{LP} is applied in equation (3.9).[12] Using this result, we can compute the gross of fees abnormal return α the GPs have to generate

[12]In general, expected returns of private equity investments will be determined by the aggregate relative risk aversion of all market participants. Thus, we implicitly assume here that the LPs relative risk a version is equal to the aggregate relative risk aversion of the market.

by the return spread,

$$\alpha = \mu_V^e - \mu_V = \mu_V^e - r_f - \eta_{LP}\, Corr_{VM}\sigma_V\sigma_M. \tag{3.10}$$

Equation (3.10) is also based on the assumption that the net present value of the LPs' position in the fund is zero in a competitive equilibrium. Thus, after fees abnormal fund returns will be equal to zero in our framework which is again consistent with the empirical findings of Jones and Rhodes-Kropf (2003) and Jegadeesh et al. (2009). More generally, we can interpret (3.10) as the minimum alpha the GPs have to generate in order to compensate investors for the fees taken. GPs can still choose a higher level of α by enhancing the expected return μ_V^e. However, we assume that this requires expending more effort for the GPs which invokes opportunity costs for them.[13]

4 Model Illustration and Analysis

4.1 Model Parameters

We first address the question of what are reasonable parameter values for our valuation model. We thereby focus on the buyout segment.

A. Fund Terms

Regarding fee terms Gompers and Lerner (1999a) and Metrick and Yasuda (2010) find that the compensation schemes for private equity funds are relatively homogeneous. For our valuation we employ the most common terms for buyout funds which are a 2 percent management fee, a carried interest level of 20 percent, and a hurdle rate of 8 percent. Additionally, we assume a typical fund lifetime of 10 years.[14]

[13] We implicitly assume here that abnormal returns α are just a consequence of managers' effort (and skill) levels. It is probable that other aspects of private equity markets also affect abnormal returns funds can achieve, such as illiquidity or a lack of competition. However, we abstract from these issues and focus on the effect managerial effort on abnormal returns here.

[14] Sahlman (1990) points out that this is the most the economic life of private equity funds which is

B. Volatility and Correlation

To estimate the volatility of the fund returns σ_V, we rely on the fact that buyout funds sometimes invest in public companies (and take them private) or in private companies that are comparable in size to small public companies. Jegadeesh et al. (2009) find that the beta of unlisted private equity funds is close to one. Therefore, we assume that the volatility σ_i of an individual investments of a fund is the same as a unit beta public stock of similar size. We set $\sigma_i = 0.6$ which resembles the small-stock volatility estimate of Campbell et al. (2001). We also need an assumption about the pair-wise correlation $Corr_{ii}$ of the individual fund investments. This pair-wise correlation is chosen to match the high end of the correlation between small-company investments in the same industry as reported in Campbell et al. (2001). This gives an estimate of $Corr_{ii} = 0.2$.

Using these figures, we can estimate the volatility of the fund returns σ_V by[15]

$$\sigma_V^2 = \frac{1}{N}\sigma_i^2(1 - Corr_{ii}) + \sigma_i^2 Corr_{ii}, \tag{4.1}$$

where N is the number of portfolio companies of the fund. Assuming a buyout fund with $N = 13$ portfolio company investments[16] this gives a volatility estimate of $\sigma_V = 0.31$.

To estimate the return correlation $Corr_{VM}$, we solve the definition of the fund beta, $\beta_V = Corr_{VM}\sigma_V\sigma_M/\sigma_M^2$, for the correlation $Corr_{VM}$. This gives

$$Corr_{VM} = \beta_V \frac{\sigma_M}{\sigma_V}. \tag{4.2}$$

Matching the results of Jegadeesh et al. (2009) with $\beta_V = 1$ and taking the index volatility reported in Campbell et al. (2001) of $\sigma_M = 0.12$ yields a reasonable correlation of $Corr_{VM} = 0.39$.

used by 72% of the funds in the sample he analyzes.

[15]See, for example, Elton and Gruber (1995), p.60-61.

[16]To generate this number the Center of Private Equity Research (CEPRES) granted us access to a comprehensive database of private equity fund investments. The 138 fully liquidated buyout funds of the database have on average invested in 18 portfolio companies with a median number of 13. We assume a median fund here and set $N = 13$.

C. Preferences and Market Price of Risk

To compute the market price of risk λ_V, we assume that GPs and LPs have a relative-risk aversion coefficient of one.[17] According to equation (3.7) this gives a market price of risk $\lambda_V = 0.05$.

D. Drawdown and Distribution Dynamics

For the dynamics of the capital drawdowns and capital distribution, we rely on the parameters estimated by Malherbe (2004) for the buyout segment: $\kappa_\delta = 8.74$, $\theta_\delta = 0.32$, $\sigma_\delta = 1.46$, $\kappa_\rho = 17.47$, $\theta_\rho = 0.20$, and $\sigma_\rho = 1.93$. This implies that in the long-run approximately 32 percent of the remaining investment capital is drawn on average and that approximately 20 percent of the current portfolio value is distributed in each year. Additionally, the high reported values for the volatilities σ_δ and σ_ρ indicate that drawdowns and distributions are fluctuating heavily over time.

Finally, the riskless rate of interest is set to a reasonable value of 5 percent p.a.

4.2 Model Illustration

Table 2 summarizes the fee values that can be computed using the set of model parameters from above. All fee values are expressed in dollars per \$100 of committed capital. The table shows that the total value of lifetime fees is \$19.31 for a fund with no catch-up and \$19.74 for a fund with a 100 percent catch-up provision. That is, total value of lifetime fee revenues amounts to around 20 percent of the committed capital of the fund. Management fees account for the largest portion of the value of the lifetime fees and are not affected by the existence of a catch-up clause. In both categories, close to four-fifth of the value of the total revenues derive from the fixed management fee and only one-fifth from the variable revenue generated by the carried interest payments. As expected, the existence of a catch-up clause increases the value of the carried interest. However, the numbers provided in Table 2 imply that the magnitude of this effect is

[17] Note that in this case LPs and GPs are both assumed to have utility of the logarithmic form.

relatively low.

[Insert Table 2 about here]

Table 2 also shows that GPs have to generate substantial abnormal returns to compensate investors for the fees taken. The GPs compensation drives a wedge between gross returns on dollars invested in a project and the net returns to fund investors. In our model, investors are perfectly competitive, so although there are positive alphas on gross fund returns, investors receive zero alphas on average net of fees. For the given fee structure, gross of fees abnormal returns amount to around 6.5 percent p.a.

4.3 Fee Value Determinants

We now explore how fee values are affected by the various model parameters. The factors determining the present value of management fees can directly be inferred from equation (3.2). Table 3 summarizes these factors and indicates their directional influence on the present value of management fees. As shown, the value increases as the fund lifetime T_l, the committed capital C, or the management fee level c_{mf} are increased. In contrast, the value decreases with higher levels of the riskless rate r_f, as this leads to higher discounting of future management fees.

[Insert Table 3 about here]

Table 4 summarizes the determinants of the present value of carried interest.[18] First, the present value of the carried interest depends on the fund terms. Similar to the present value of management fees, the present value of the carry increases as the fund lifetime T_l or the committed capital C is increased. The carried interest entitles the

[18]For simplicity, we assume here that no management fees are paid to analyze the impact of the various model parameters on the value of the carried interest. Note that the management fees decrease investment capital by the mechanism shown in equation (**??**) and therefore will also affect carried interest payments.

GPs a percentage of the capital gains of the fund equal to the carried interest level c_{ci} if the fund's return exceeds the hurdle rate h. Therefore, the value increases (decreases) with an increasing carried interest level c_{ci} (hurdle rate h). Second, the present value of the carry is influenced by the parameters governing the dynamics of the drawdown and distribution rate. The main model parameters affecting the carried interest value here are the long-term drawdown rate θ_δ and the long-term distribution rate θ_ρ. A higher long-term drawdown rate θ_δ increases the value of the carried interest. This holds as a higher parameter θ_δ on average leads to a faster drawdown schedule. A faster drawdown schedule in turn results in earlier carried interest payments on average. Through the effect of a lower discounting of earlier carried interest payments this results in higher carried interest values. Conversely, increasing the long-term distribution rate θ_ρ decreases the value of the carried interest. This holds because a higher level of θ_ρ decreases the average period over which the capital is tied up in portfolio companies which also decreases the probability of fund returns that exceed the hurdle rate. The impact of the other model parameters on the carry value is only very limited. Increasing the volatility parameter σ_δ (σ_ρ) increases (decreases) the carry value. The impact of a variation of the speed of adjustment parameters κ_δ and κ_ρ is about the same relative magnitude. However, the speed of adjustment parameters have an opposite effect on the carry value which can be explained by the fact that high parameters κ_δ and κ_ρ can absorb some of the volatility of the drawdown and distribution rate. Third, carry value is influenced by the asset value dynamics and the riskless rate of interest. Because of the option-like structure of the carry a higher asset volatility σ_V increases the probability of high carried interest payments and thus increases the value of the carried interest. Increasing the market price of risk λ_V decreases the carry value. This holds because a higher market price of risk implies a higher systematic risk of the fund which leads to a higher discounting of future carry payments. Finally, the value of the carried interest decreases with higher levels of the riskless rate r_f.

[Insert Table 4 about here]

Analyzing determinants of fee value naturally leads to the question how a GP can increase the value of his position. As already pointed out, compensation schemes and fund lifetimes in the private equity asset class are relatively homogeneous. This implies that the manager has only little control over these variables. However, managers have some control over the other main determinants of carry value, the fund's drawdown and distribution policy and the volatility of assets. Regarding the drawdown and distribution policy, our model implies that managers can increase the carry value by increasing the long-term drawdown rate θ_δ and decreasing the long-term distribution rate θ_ρ. This results in a longer average time over which capital is tied up in the fund which makes high carry payment more likely. However, the potential to increase fee value by altering the drawdown and distribution policy of the fund is limited by the fixed lifetime of the fund and by the fact that it might take some time to find attractive portfolio companies. In contrast, the manager is in full control of the risk of the projects he chooses. Our results here imply that the manager has an incentive to increase the volatility to enhance carry value. We will explore this issue more detail in the following.

4.4 Risk Taking Incentives

We first study risk-taking incentives for GPs that raise only a single fund. Subsequent, we extend the analysis to the more realistic setting of GPs that raise multiple funds in sequence. Finally, we analyze incentives of GPs to alter risk during the life of a fund.

A. Single Fund

The previous analysis has shown that the option-like carried interest position increases in value as the fund volatility rises. Thus, it seems that the typical compensation scheme of private equity funds does encourage excessive risk taking by the GPs. This incentive to increase risk is illustrated in Figure 1 that shows that total fee value is strictly increasing in asset volatility σ_V of the fund. In the figure we also allow for the possibility that the manager has some control over the (gross of fee) abnormal returns α of the fund. As can

be seen, fee values are also strictly increasing abnormal returns α. Thus, the option-like compensation scheme also gives GPs an incentive to work hard if we assume that they can enhance abnormal returns α via expending more effort.[19]

[Insert Figure 1 about here]

The analysis behind Figure 1 is based on the implicit assumption that the GPs relative risk aversion η_{GP} is equal to the aggregate relative risk aversion η_{LP} of the market. In this case, the risk-neutral drift coefficient of the fund value dynamics (3.6) used to evaluate equation (3.3) is equal to $\alpha + r_r$, and is thus independent of the GPs relative risk aversion η_{GP}. In a more general setting with different levels of risk aversion, i.e. $\Delta\eta = \eta_{GP} - \eta_{LP} \neq 0$, the risk-neutral drift is given by $\alpha + r_f - \Delta\eta Corr_{VM}\sigma_M\sigma_V$. Because GPs hold significant amounts of wealth in the form of fund fee compensation and are unable to monetize these positions, we assume that they display a higher level of risk-aversion, i.e. $\Delta\eta > 0$ holds. In this case, the risk neutral drift will be decreasing in the asset volatility σ_V. As the diffusion coefficient in (3.6) is unaffected by $\Delta\eta$, this generally still leads to a situation in which fees values are still increasing in the asset volatility σ_V, but at a slower rate than shown in Figure 1.[20]

Two implications can be drawn from the previous analysis. The first implication is that the option-like compensation scheme gives GPs an incentive to engage in activities that generate high abnormal returns, which is precisely what benefits the LPs. The second implication, however, is that the compensation contract generally encourages excessive risk taking by the GPs, particularly if they display a low relative risk aversion. This second point is problematic for the LPs because it might encourage GPs to take undue risks. In some situations, for example, it will sense for GPs to accept high risk

[19]Note that GPs in general must invest significant time and effort (including board meetings, meeting with management/customers/suppliers, understanding the market, etc.) to help portfolio companies increase in value (see e.g. Sahlman (1990)). However, abnormal returns of private equity funds must not necessarily only be the consequence of GPs' effort levels. Probably other aspects of the private equity market also drive abnormal returns, such as illiquidity, agency problems, or a lack of competition.

[20]Note that fee values can even decrease with the asset volatility σ_V when $\Delta\eta$ gets very large. However, for reasonable values of $\Delta\eta$ fee values will be an increasing function in asset volatility σ_V.

but negative-net-present value investments because doing so increases the value of their carry option.

B. Multiple Funds in Sequence

The previous analysis was limited in that it considered only a single fund. In a more realistic setting, GPs do consider not only fee compensation from their current fund, but also potential subsequent compensation based on follow-on funds. In this case, GPs will take into account that risk-taking behavior and effort level also affect their ability to raise funds in the future. We now extend the setting to GPs that aim to raise multiple funds in sequence and maximize the net present value of their lifetime fee income.

[Insert Figure 2 about here]

Figure 2 illustrates the extended model setting. For simplicity, we assume that all subsequent funds have the same lifetime T_l and that all model parameters are the same for all funds. In particular, GPs chose the same level of abnormal return α and the same volatility σ_V in each fund. For this reason, the present value of fee income is the same for each fund at fund initiation and is denoted by V_0^{GP}. Figure 2 shows that the probability of raising a follow-on fund is q in each step. In line with empirical evidence, the ability of the fund manager to raise a follow-on fund in our model depends on the performance of his current fund.[21] We assume that GPs can only raise follow-on funds if the final performance of the current fund exceeds some defined threshold b. In this case the probability of raising a follow-on fund is equal to the probability that the internal rate of return at fund liquidation T_l exceeds the level b, i.e.

$$q = Prob(IRR_{T_l} \geq b). \tag{4.3}$$

We consider GPs that aim to raise m funds in sequence. If q denotes the risk-neutral

[21] Empirical studies have shown that there exists a significant relationship between investment performance and future fundraising in that more successful GPs find it easier to raise capital for follow-on funds. See e.g. Gompers and Lerner (1998), Kaplan and Schoar (2005), or Hochberg et al. (2010).

probability, the net present value of the lifetime fee income of the GPs is given by

$$NPV_{0,m}^{GP} = V_0^{GP} + V_0^{GP}\frac{q}{(1+r)^{T_l}} + V_0^{GP}\frac{q^2}{(1+r)^{2T_l}} + \ldots + V_0^{GP}\frac{q^m}{(1+r)^{mT_l}}. \tag{4.4}$$

If we assume an infinite number of funds, i.e. $m \to +\infty$, equation (4.4) simplifies to

$$NPV_0^{GP} = \lim_{m\to+\infty} NPV_{0,m}^{GP} = V_0^{GP}\frac{1}{1-\frac{q}{(1+r)^{T_l}}}. \tag{4.5}$$

Figure 3 illustrates the present value of fee income NPV_0^{GP} as a function of fund asset volatility σ_V and (gross of fee) abnormal returns α for different levels of the return threshold b. A comparison between Figure 3 and Figure 1 shows that the case with multiple funds in sequence creates a very different incentive structure than the single fund case. In particular, lifetime fee values are no longer strictly increasing in the asset volatility σ_V. Now, it depends on the level of abnormal return α whether the fee value is increasing or decreasing in σ_V. This result becomes reasonable if we consider α as a measure for the skill of the GPs. For low skilled GPs, higher volatilities increase carry payments and the probability that the final fund performance exceeds the return threshold b. Thus, V_0^{GP} and q in equation (4.5) are both increasing in σ_V. This gives low skilled GPs a high incentive to increase risk. For high skilled GPs, V_0^{GP} is also increasing in σ_V but the probability q is decreasing in σ_V, as higher volatilities then increase the probability of low returns. As α grows, the second effect will dominate the first and the present value of fee income NPV_0^{GP} will eventually decrease with the volatility σ_V. That is, high skilled GPs will even have an incentive to decrease fund risk. Figure 3 also studies the effect of different levels of the return threshold b. It shows that the effect of increasing b is that the region over which fee values are increasing in the volatility σ_V gets larger. This is reasonable because a high return threshold will give even high skilled investors an incentive to gamble by increasing fund risk.[22]

[22]Similar to the analysis in the single fund case, Figure 3 was derived under the simplifying assumption that the GPs relative risk aversion is equal to the aggregate relative risk aversion of the market. Introducing a higher level of risk aversion for the GPs will result in lower incentives to gamble,

[Insert Figure 3 about here]

Overall, our results show that the pressure to raise follow-on funds that stems from the finite lifetime of private equity funds creates a unique incentive structure for the GPs that is very different from the single fund case. Put in option pricing terminology, GPs are essentially equipped with two kinds of option contracts on the same underlying in the case with multiple funds in sequence. The first option, as already noted, is the carry option that shares some characteristics with a standard call option on the performance of the fund. The second option is to receive compensation from follow-on funds which shares some characteristics with a binary cash-or-nothing call option on the performance of the fund. This holds because GPs receive the present value of fees of all follow-on funds in the case that the performance of their current fund exceeds some threshold level and nothing otherwise. The unique incentive structure of GPs is then created by the fact that both option contracts can react differently to increases in fund volatility. The value of the carry option is always strictly increasing in fund volatility. In contrast, the value of the option to receive compensation of follow-on funds decreases with volatility if the GPs skill is sufficiently high. As the second effect generally offsets the first, the combination of both option-like contracts limits risk taking incentives of GPs.

Why is this complex incentive structure necessary in the private equity segment to limit risk taking? Probably the most important reason has to do with agency problems between GPs and LPs. In theory, GPs and LPs can negotiate the risk strategy before the fund is set up. However, ex-post, after the fund has been initiated, GPs will have an incentive to increase risk taking above the negotiated level to enhance fee value. This holds because LPs will usually have very limited abilities to observe the GPs risk taking over the life of a fund. Even if LPs could perfectly observe the GPs risk taking their possibilities to react to the GPs behavior are very limited. In particular, withdrawal from the partnership before the termination date is prohibited by contractual arrangements (see Sahlman (1990)). Therefore, some external mechanics must be installed that limits

particularly for GPs with a relatively low level of skill.

GPs risk taking appetite. As shown, this is achieved by the pressure to raise follow-on funds which stems from the finite lifetime of private equity funds. While this limits GPs risk taking it still leaves them with an incentive to generate high abnormal returns. However, as shown, the mechanism is not perfect in the sense that it still gives low skilled investors an incentive to gamble. Solving this by contractual arrangements is difficult because LPs abilities to distinguish whether a high fund performance is the results of high skill or mere luck is typically limited.

Our results provide a number of testable predictions that are consistent with carefully documented empirical facts. First, our results imply that inexperienced and low skilled GPs chose higher levels of risk. This is consistent with the results of Ljungqvist et al. (2008) that show that younger funds invest in riskier deals and that risk taking is reduced as they become more experienced. Second, the model implies that risk taking will also depend on the state of the private equity market. If the return threshold b increases in tough times, average risk taking of private equity funds should increase. As this should go along with higher average returns, our model predicts a countercyclical investment performance of private equity funds. This prediction is consistent with studies that have found evidence that suggests such a countercyclical investment performance in both the buyout (see Kaplan and Stein (1993)) and the venture capital market (see Gompers and Lerner (2000)).

C. Intra-Fund Variation of Risk

Until so far, we have assumed that GPs cannot change the volatility σ_V during the life of a fund. We now turn to the question if GPs have an incentive to alter risk taking during fund lifetime in response to the past performance of the fund. In a general context, Carpenter (2000) points out that an option-like compensation contract encourages managers to dynamically adjust volatility as the asset value changes. Her model shows that it is optimal for the manager to moderate portfolio risk as the asset value grows large and to increase portfolio risk as the asset value tends to zero.

This general problem is even gravitated in the private equity area because past fund performance both has an impact of on the value of the performance related compensation and on the manager's ability to raise follow-on funds. If the current fund is performing poorly, the carried interest option is deep out of the money and the GP's probability to raise a follow-on fund is low. In this case, there is a strong incentive for excessive risk taking, as higher volatilities both increase carry value and the probability to raise a follow-on fund. The further out of the money the carry option and the shorter the time to maturity, the more the GPs is willing to gamble. Yet, there is not always an incentive for greater risk taking. On the other hand, if the fund performs well, the carried interest option is deep in the money and the probability of being able to raise a follow-on fund is high. In this case, GPs will have an incentive to damp down the risk of the fund, as this enables them to lock-in the carry value and to increase the probability of being able to raise a follow-on fund. So, the structure of private equity fund leaves some incentive problems during the fund lifetime.

Interestingly, these incentive problems are mostly resolved in standard private equity contracts. A natural way for a GPs to react to the problems described above would be to keep a certain amount of the investment capital of their funds undrawn until they learn about the performance of the investments already in place. Then, if the fund performs poorly (well), GPs could invest these undrawn amounts into high (low) risk projects. However, this is generally not possible because standard contractual arrangements allow GPs only to draw down capital over a pre-specified commitment period, which is usually only three to five years long. During this time period, GPs will in most cases only have very limited ability to predict the final performance of their investments in place. Similarly, the problems described above might give GPs an incentive to reinvest the proceeds of their liquidated investment according to the past performance of the fund. Note that this is also not possible because private equity fund managers by standard contractual arrangements directly have to pass through all proceeds of the fund (whether in the form of equity or cash) to their investors (see Lerner and Schoar (2004)).

In the light of these discussions, we can give an interesting explanation for the seemingly odd contractual arrangements that GPs are only allowed to draw down capital over a limited commitment period and are not allowed to reinvest proceeds of a fund. Particularly, the second one does not seem to make sense for GPs that have access to attractive investment opportunities. But, as discussed, both contractual features are means of limiting risk shifting behavior of fund managers during the lifetime of a fund.

4.5 Diversification

The previous analysis has shown that the compensation scheme of private equity fund managers does not necessarily result in an incentive to increase risk taking. High skilled GPs can even enhance their lifetime fee values by reducing overall fund risk. If an incentive for risk taking is not the general effect, what is the rationale behind the option-like compensation contract that is standard in the industry? Our model can provide an interesting explanation for this. We can show that that the contract encourages GPs to search for investments that have a low correlation with the market returns and to have a high level of idiosyncratic risk of a fund. We first illustrate this result and then, we explain why this makes sense from the perspective of the fund investors.

To illustrate these points, again assume that GPs and LPs have different levels of risk-aversion η_{GP} and η_{LP}, respectively. Because GPs hold significant amounts of wealth in the form of fund fee compensation and are unable to monetize these positions, we assume that they display a higher level of risk-aversion, i.e. $\Delta\eta = \eta_{GP} - \eta_{LP} > 0$ holds. To show the effect of this on the carry value, we consider a fixed level of total risk σ_V^*. For this level of total risk σ_V^*, the risk-neutral drift coefficient of the fund value dynamics (3.6) used to evaluate equation (3.3) is given by

$$\alpha + r_r - \Delta\eta Corr_{VM}\sigma_M\sigma_V^*. \tag{4.6}$$

This equation shows that the drift coefficient increases as the correlation between fund

and market returns $Corr_{VM}$ decreases. A higher drift coefficient in turn leads to higher carry values because the diffusion coefficient in (3.6) depends only on the fixed volatility σ_V^*. This implies that GPs should look for a very low return correlation $Corr_{VM}$ to increase carry value. Additionally, as total risk σ_V^* is fixed here, this also implies that this kind of contract gives GPs an incentive for a low systematic risk and a high idiosyncratic risk component of the fund. Interestingly, both effects get more pronounced as $\Delta\eta$ increases. That is, the incentive for the GPs to decrease $Corr_{VM}$ increases with their relative risk aversion η_{GP}.

Why does this incentive structure make sense for the LPs? First, a major reason why LPs invest in private equity funds is that they seek additional diversification for their overall portfolios, not exposure to market risk. Thus, it makes sense that they want the return correlation of the fund $Corr_{VM}$ to be as low as possible to maximum diversification benefits from their private equity exposure. Second, GPs must invest significant time and effort in managing portfolio companies. Therefore, they will only be able to manage a limited number of investments.[23] Additionally, their expertise may be limited to a particular sector or industry, which means that funds are exposed to substantial sector risk. Even if GPs can diversify across the entire private equity universe, funds may not be fully diversified because projects may contain a correlated idiosyncratic risk component. For these reasons, private equity funds are generally exposed to significant levels of idiosyncratic risk. In the light of this discussion, it makes sense to give GPs an incentive to have a high idiosyncratic risk component of the fund. This encourages GPs to focus on actively managing a portfolio of a limited amount of investments, rather than on building up large diversified portfolios that are managed passively. Thus, it prevents that GPs have may have the incentive to spend too little time with any one investment in the attempt to manage more projects.[24]

[23]For example, Gompers and Lerner (1999a) point out GPs typically invest in at most two dozen firms over thee fund's commitment period.

[24]One could argue here that a high idiosyncratic risk of the funds is unfavorable for LPs. However, note that usually private equity investors have enough wealth outside the fund that they are still well-diversified. Therefore, high idiosyncratic risk of their private equity exposure will only have little effect on the diversification levels of their overall portfolios.

5 Conclusion

This paper introduces a risk-neutral option-pricing approach to the valuation of private-equity fund fees. This approach allows us to study determinants of private equity fund fee value and to analyze incentives generated by the standard compensation schemes. The risk-taking incentives we find are much richer than standard option pricing theory suggests. This results from our ability to show that the finite lifetime of private equity funds and the related pressure to raise follow-on funds introduces another nonlinearity into GPs compensation. Effectively, GPs are equipped with a second option-like position that shares some characteristics with a binary cash-or-nothing call in case they aim to raise multiple funds in sequence. In this case, it depends on the GPs level of skill whether fee value is increasing or decreasing in asset volatility. The general implication of this is that analyzing risk-taking incentives in models with multiple evaluation periods can lead to very different results than what can be generated in a one-period setting. As the manager's behavior clearly depends on his whole compensation package, models with multiple evaluation periods should give a more realistic picture. We think this indicates a fruitful direction in which to proceed in analyzing similar incentive issues in the future.

From a policy perspective, it is clear that the typical private equity compensation contract can result in high risk-taking incentives for low skilled GPs, especially if they also display a low relative risk aversion. It is important that fund investors understand this structure and carefully screen managers accordingly. Additionally, our results suggest that even high skilled GPs will have an increasing incentive to gamble when market conditions get tougher, i.e. when they assume that the return threshold they have to exceed to raise a follow-on fund has increased substantially. Controlling such risk-taking incentives is difficult for LPs. However, contractual limits and verifiable risk management constraints should help.

A Equilibrium Framework

We consider a partial-equilibrium framework to solve for the expected rate of return μ_V. The equilibrium framework is qualitatively similar to the one developed in Berk and Green (2004) in the context of mutual funds. The basic idea is that investors supply their capital competitively but fund management skill is a scarce resource. In this case investors' expected excess returns (net of fees) must equal zero in equilibrium.

To derive the equilibrium condition, let V_t^{LP} denote value of the LPs position at time t. As the LPs receive capital distributions of the fund net of carried interest payments, it must hold that

$$V_t^{LP} = E_t^{\mathbb{Q}} \left[\int_t^{T_l} e^{-r_f(u-t)} (dR_u - dCI_u) \right]. \tag{A.1}$$

In order to establish their positions the LPs have to incur costs. These costs include future capital drawdowns and management fees. The present value of the outstanding costs of the LPs at time t, C_t^{LP}, is thus defined by

$$C_t^{LP} = E_t^{\mathbb{Q}} \left[\int_t^{T_l} e^{-r_f(u-t)} (dD_u + dMF_u) \right]. \tag{A.2}$$

In a partial-equilibrium, the LPs will only commit capital to the fund at $t = 0$ as long as the value of their position at least covers the costs, i.e. $V_0^{LP} \geq C_0^{LP}$. Stated differently, the LPs will only be willing to invest in the fund if the net present value of their position is non-negative. Similar to Berk and Green (2004), we assume a competitive market for private equity investment, where the GPs capture all rents, such that $V_0^{LP} = C_0^{LP}$ holds exactly. This implies a zero net present value for the LPs at fund initiation.[25] Using

[25] In particular, we assume that the managers are in possession of scarce skills and are able to adjust fund terms such that they capture all of the rents from these skills.

this assumption, we can develop the following equilibrium condition

$$E^{\mathbb{Q}}\left[\int_0^{T_l} e^{-r_f u}(dD_u + dMF_u)\right] = E^{\mathbb{Q}}\left[\int_0^{T_l} e^{-r_f u}(dR_u - dCI_u)\right]. \tag{A.3}$$

That is, in equilibrium it must hold that the present value of lifetime management fees and capital drawdowns (LHS) is equal to the present value of the lifetime capital distributions net of carried interest payments (RHS). Rearranging this equation yields

$$E_0^{\mathbb{Q}}\left[\int_0^{T_l} e^{-r_f u}(dR_u - dD_u - dMF_u - dCI_u)\right] = 0, \tag{A.4}$$

which states that the net present value of the fund's net cash flows after management fees and carried interest payments must be zero at fund initiation in equilibrium.

It is this condition that we use to derive the equilibrium expected rate of return μ_V^e. Thereby, the correct expected rate of return can be found by simply looking for the value of μ_V that makes this equation to hold. A solution to this problem can only be found by using numerical techniques. We use a Monte Carlo simulation of a discrete-time approximation of (A.4) to solve for the expected rate of return. The details of this simulation approach are outlined in Appendix B.

B Monte Carlo Simulation

In this appendix we present a numerical solution of valuation equation (3.3) and equilibrium condition (3.8) by Monte Carlo simulation of a discrete-time approximation of the stochastic cash flow model presented in Section 2.1.

In order to implement the Monte Carlo simulation, we divide the time interval $[0, T_l]$ into K discrete intervals each of length Δt. Then, we simulate all relevant quantities at equidistant time points $t_k = k\Delta t$, where $k = 1, \ldots, K$ and $K = T_l/\Delta t$ holds.

Using these definitions, the dynamics of the capital drawdowns, (2.1), can be represented in discrete-time by[26]

$$\Delta D_{k+1} = \delta_{k+1} U_k \Delta t. \tag{B.1}$$

Similarly, it holds that an appropriate discrete-time specification of the dynamics of the capital distributions, (2.4), is given by

$$\Delta R_{k+1} = \rho_{k+1} V_k \Delta t, \quad \text{if} \quad k+1 < K, \quad \text{and} \quad \Delta R_K = V_K. \tag{B.2}$$

An appropriate scheme for approximating the risk-neutral dynamics of the drawdown and distribution rate, equations (3.4) and (3.5), is the *Milstein scheme*.[27] Applying this

[26] For simplicity, we assume here that the commitment period T_c is equal to the legal lifetime of the fund T_l, i.e., $T_c = T_l$ holds.

[27] With its second-order Taylor approximation, the *Milstein scheme* here provides a better approximations of the SDEs compared to the standard *Euler scheme* which is only based on a first-order Taylor approximation. For an arbitrary SDE, $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$, the Milstein approximation takes the form

$$X_{k+1} = X_k + \mu(X_k)\Delta t + \sigma(X_k)\sqrt{\Delta t}\ \epsilon_{k+1} + \frac{1}{2}\sigma(X_k)\sigma'(X_k)\Delta t[(\epsilon_{k+1})^2 - 1],$$

where ϵ_{k+1} is a standard normal variable and $\sigma'(X_k) = d\sigma(X_k)/dX_k$ holds. For more details on how to approximate SDEs in discrete-time, see e.g. Glasserman (2003) or Kloeden and Platen (1999).

scheme, it turns out

$$\delta_{k+1} = \delta_k + \kappa_\delta(\theta_\delta - \delta_k)\ \Delta t + \sigma_\delta\ \sqrt{\delta_k \Delta t}\ \epsilon_{\delta,k+1} + \frac{1}{4}\sigma_\delta^2 \Delta t[(\epsilon_{\delta,k+1})^2 - 1], \tag{B.3}$$

$$\rho_{k+1} = \rho_k + \kappa_\rho(\theta_\rho - \rho_k)\ \Delta t + \sigma_\rho\ \sqrt{\rho_k \Delta t}\ \epsilon_{\rho,k+1} + \frac{1}{4}\sigma_\rho^2 \Delta t[(\epsilon_{\rho,k+1})^2 - 1], \tag{B.4}$$

where $\epsilon_{\delta,1}, \epsilon_{\delta,2}, \ldots, \epsilon_{\delta,K}$ and $\epsilon_{\rho,1}, \epsilon_{\rho,2}, \ldots, \epsilon_{\rho,K}$ are *i.i.d.* sequences of standard normal variables that are assumed to be uncorrelated with each other.

Finally, approximating the risk-neutral dynamics of the investment value, (3.6), using the *Milstein scheme* gives

$$\begin{aligned} V_{k+1} =& V_k[1 + (\mu_V + \lambda_V \sigma_V)\Delta t] + \sigma_V V_k \epsilon_{V,k+1}\sqrt{\Delta t} + \frac{1}{2}\sigma_V^2 V_k \Delta t[(\epsilon_{V,k+1})^2 - 1] \\ & - \Delta R_{k+1} + \Delta D_{k+1}, \end{aligned} \tag{B.5}$$

where $\epsilon_{V,1}, \epsilon_{V,2}, \ldots, \epsilon_{V,K}$ is a third *i.i.d.* sequence of standard normal variables that is assumed to be uncorrelated with the sequences $\epsilon_{\delta,1}, \epsilon_{\delta,2}, \ldots, \epsilon_{\delta,K}$ and $\epsilon_{\rho,1}, \epsilon_{\rho,2}, \ldots, \epsilon_{\rho,K}$.

Using these discrete-time specifications, the risk-neutral carried interest dynamics (assuming no catch-up) can be approximated by

$$\Delta CI_k = c_{ci} \max\{\Delta R_k - \Delta D_k - \Delta MF_k, 0\} 1_{\{IRR_k > h\}}, \tag{B.6}$$

where IRR_k is a solution to

$$\sum_{s=1}^{k} \frac{\Delta R_s - \Delta D_s - \Delta MF_s}{(1 + IRR_k)^s} = 0. \tag{B.7}$$

To numerically solve equation (3.3), consider a Monte Carlo sampling experiment composed of M independent replications of the discrete-time approximations (B.1) to (B.7). Let $\Delta R_{k,i}$ ($\Delta D_{k,i}$) denote the kth observation of the capital distributions (capital drawdowns) in the mth replication. Similarly, $IRR_{k,i}$ denotes the kth observation of the

internal rate of return in the mth replication. If the number of simulation iterations M is considerably large, the value of the outstanding carried interest with no catch-up in equation (3.3) can be approximated by computing

$$V_k^{CI} = \frac{1}{M} \sum_{i=1}^{M} \sum_{s=k}^{K} \left[\frac{c_{ci} \max\{\Delta R_{s,i} - \Delta D_{s,i} - \Delta MF_s, 0\} 1_{\{IRR_{s,i} > h\}}}{(1 + r_f \Delta t)^{s-k}} \right]. \tag{B.8}$$

Note that the same approach can also be used to determine the value of the outstanding carried interest with a 100 percent catch-up provision.

Similarly, we can express the net present value of the LPs' fund position that appears on the LHS of equilibrium condition (3.8) by the discrete-time approximation

$$NPV_0^{LP} = \frac{1}{M} \sum_{i=1}^{M} \sum_{s=0}^{K} \left[\frac{\Delta R_{s,i} - \Delta D_{s,i} - \Delta MF_{s,i} - \Delta CI_{s,i}}{(1 + r_f \Delta t)^{s}} \right]. \tag{B.9}$$

Using this approximation, we can solve equilibrium condition (3.8) for the expected rate of return μ by a numerical optimization of

$$\min_{\mu} \left\{ \sum_{i=1}^{M} \sum_{s=0}^{K} \left[\frac{\Delta R_{s,i} - \Delta D_{s,i} - \Delta MF_{s,i} - \Delta CI_{s,i}}{(1 + r_f \Delta t)^{s}} \right] \right\}^2, \tag{B.10}$$

which is done for a given set of the other model parameters. Note that throughout this paper, we choose $M = 500,000$ simulation iterations to evaluate equations (B.8) and (B.10). In addition, we set $\Delta t = 0.25$, i.e., all simulation are carried out on a quarterly basis.

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Tables and Figures

Table 1: Carried Interest Calculation

This table illustrates the carried interest calculation for a $100M fund with a carried interest level of 20 percent, a hurdle rate of 8 percent, and a lifetime of ten years. The calculation is shown for a fund with no catch-up clause and fund with a catch-up clause of 100 percent.

Year	1	2	3	4	5	6	7	8	9	10	Total
Cash Flows	-50	-30	-10	-10	30	50	60	50	40	20	150
IRR (in % p.a.)	-100	-100	-100	-100	-33	-6	8	14	17	18	18
Carried Interest (No Catch-Up)	0	0	0	0	0	0	0	10	8	4	22
Carried Interest (With Catch-Up)	0	0	0	0	0	0	0	18	8	4	30

Table 2: Estimated Fee Values and Abnormal Returns

This table summarizes the outputs of the fee valuation. Fee values are expressed in dollars per $100 of committed capital. The fee terms employed for the calculations are a 2 percent management fee, a carried interest level of 20 percent, and a hurdle rate of 8 percent. The table also shows (gross of fees) abnormal fund returns necessary to compensate LPs for the fees taken. These abnormal returns are given in percent p.a. Calculations are shown for a fund with no catch-up clause and fund with a catch-up clause of 100 percent.

	Management Fee Value	Carried Interest Value	Total Fee Value	Abnormal Return
No Catch-up	15.86	3.45	19.31	6.30%
With Catch-up	15.86	3.88	19.74	6.55%

Table 3: Factors Determining Management Fee Value

This table summarizes the factors determining the present value of management fees at fund inception date, V_0^{MF}. The column labeled direction indicates how the present value of management fees changes if the corresponding parameter is increased.

Parameter		Direction
Fund lifetime	T_l	increases
Committed capital	C	increases
Management fee level	c_{mf}	increases
Riskless rate	r_f	decreases

Table 4: Factors Determining Carried Interest Value

This table summarizes the factors determining the present value of carried interest at fund inception date, V_0^{CI}. The column labeled direction indicates how the present value of carried interest changes if the corresponding parameter is increased. For parameters with only a small impact on the carried interest value, the direction is shown in brackets. Note that the relationships shown in this table hold for carried interest payments with and without a catch-up clause.

Parameter		Direction
Fund lifetime	T_l	increases
Committed capital	C	increases
Carried interest level	c_{ci}	increases
Hurdle rate	h	decreases
Speed of adjustment drawdown rate	κ_1	(decreases)
Long-term drawdown rate	θ_1	increases
Volatility drawdown rate	σ_1	(increases)
Speed of adjustment distribution rate	κ_2	(increases)
Long-term distribution rate	θ_2	decreases
Volatility distribution rate	σ_2	(decreases)
Market price of risk	λ_V	decreases
Asset volatility	σ_V	increases
Riskless rate	r_f	decreases

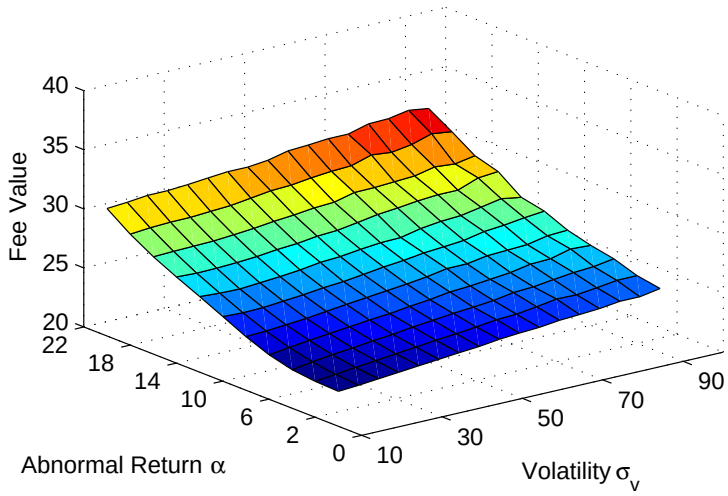

Figure 1: Fee Value of a Single Fund as a Function of Abnormal Return α (in % p.a.) and Volatility σ_V (in % p.a.)

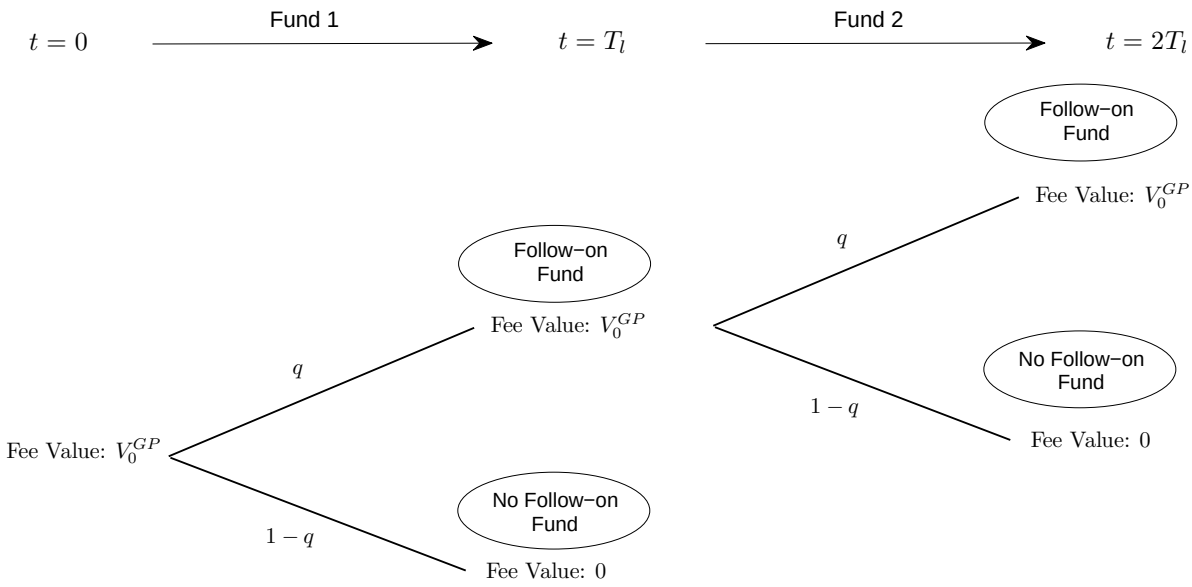

Figure 2: Model Setting with Multiple Funds in Sequence

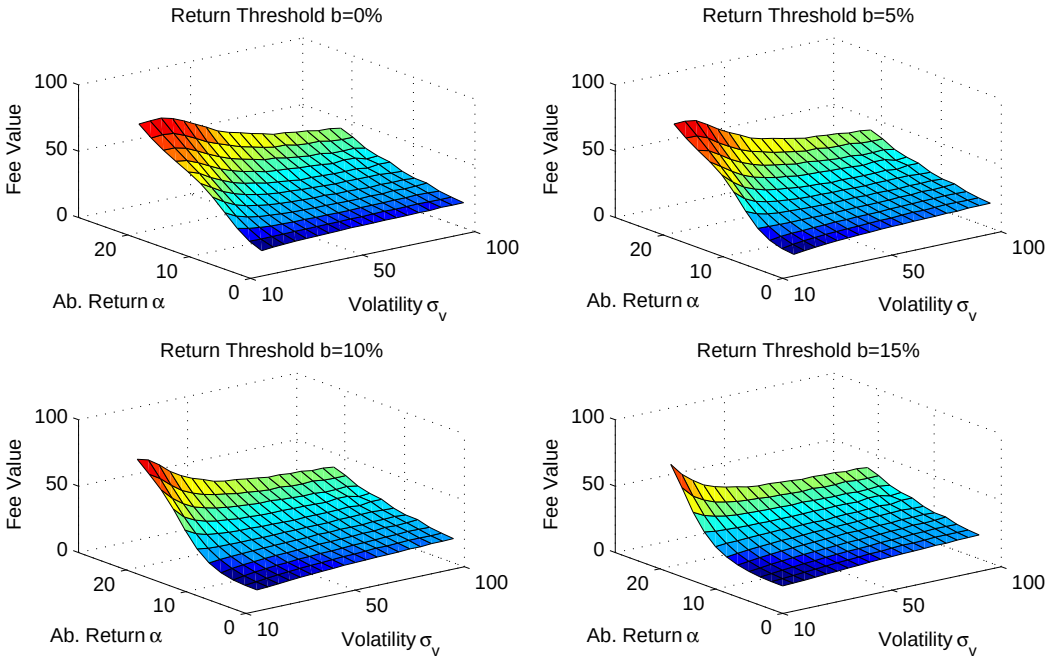

Figure 3: Fee Value of Multiple Funds in Sequence as a Function of Abnormal Return α (in % p.a.) and Volatility σ_V (in % p.a.)